

# ON ZERO-SUM MAGIC GRAPHS AND THEIR NULL SETS

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ABSTRACT. For any  $h \in \mathbb{N}$ , a graph  $G = (V, E)$  is said to be  $h$ -magic if there exists a labeling  $l : E(G) \rightarrow \mathbb{Z}_h - \{0\}$  such that the induced vertex labeling  $l^+ : V(G) \rightarrow \mathbb{Z}_h$  defined by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. When this constant is 0 we call  $G$  a zero-sum  $h$ -magic graph. The null set of  $G$  is the set of all natural numbers  $h \in \mathbb{N}$  for which  $G$  admits a zero-sum  $h$ -magic labeling. A graph  $G$  is said to be uniformly null if every magic labeling of  $G$  induces zero sum. In this paper we will identify the null sets of the generalized theta graphs and will introduce a class of uniformly null magic graphs.

## 1. INTRODUCTION

In this paper all graphs are connected, finite, simple, and undirected. For an abelian group  $A$ , written additively, any mapping  $l : E(G) \rightarrow A - \{0\}$  is called a *labeling*. Given a labeling on the edge set of  $G$  one can introduce a vertex set labeling  $l^+ : V(G) \rightarrow A$  by

$$l^+(v) = \sum_{uv \in E(G)} l(uv).$$

A graph  $G$  is said to be  $A$ -*magic* if there is a labeling  $l : E(G) \rightarrow A - \{0\}$  such that for each vertex  $v$ , the sum of the labels of the edges incident with  $v$  are all equal to the same constant; that is,  $l^+(v) = c$  for some fixed  $c \in A$ . In general, a graph  $G$  may admit more than one labeling to become  $A$ -magic; for example, if  $|A| > 2$  and  $l : E(G) \rightarrow A - \{0\}$  is a magic labeling of  $G$  with sum  $c$ , then  $\lambda : E(G) \rightarrow A - \{0\}$ , the *inverse labeling* of  $l$ , defined by  $\lambda(uv) = -l(uv)$  will provide another magic labeling of  $G$  with sum  $-c$ . A graph  $G = (V, E)$  is called *fully magic* if it is  $A$ -magic for every abelian group  $A$ . For

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example, every regular graph is fully magic. A graph  $G = (V, E)$  is called *non-magic* if for every abelian group  $A$ , the graph is not  $A$ -magic. The most obvious example of a non-magic graph is  $P_n$  ( $n \geq 3$ ), the path of order  $n$ . As a result, any graph with a path pendant of length at least two would be non-magic. Here is another example of a non-magic graph: Consider the graph  $H$  in Figure 1. Given any abelian group  $A$ , a potential magic labeling of  $H$  is illustrated in that figure. The condition  $l^+(u) = l^+(v)$  implies that  $7x + y = 7x = y = z$  or  $z = 0$ , which is not an acceptable magic labeling. Thus  $H$  is not  $A$ -magic.

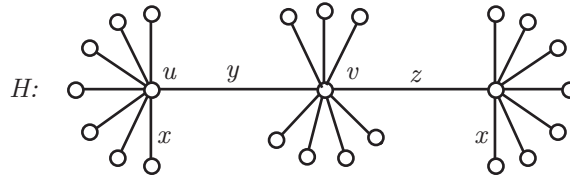


FIGURE 1. An example of a non-magic graph.

Certain classes of non-magic graphs are presented in [1].

The original concept of  $A$ -magic graph is due to J. Sedlacek [14, 15], who defined it to be a graph with a real-valued edge labeling such that

- (1) distinct edges have distinct nonnegative labels; and
- (2) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Jenzy and Trenkler [3] proved that a graph  $G$  is magic if and only if every edge of  $G$  is contained in a (1-2)-factor.  $\mathbb{Z}$ -magic graphs were considered by Stanley [16, 17], who pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. Recently, there has been considerable research articles in graph labeling, interested readers are directed to [2, 18]. For convenience, the notation 1-magic will be used to indicate  $\mathbb{Z}$ -magic and  $\mathbb{Z}_h$ -magic graphs will be referred to as  $h$ -magic graphs. Clearly, if a graph is  $h$ -magic, it is not necessarily  $k$ -magic ( $h \neq k$ ).

**Definition 1.1.** For a given graph  $G$  the set of all positive integers  $h$  for which  $G$  is  $h$ -magic is called the integer-magic spectrum of  $G$  and is denoted by  $IM(G)$ .

Since any regular graph is fully magic, then it is  $h$ -magic for all positive integers  $h \geq 2$ ; therefore,  $IM(G) = \mathbb{N}$ . On the other hand, the graph  $H$ , Figure 1, is non-magic, hence  $IM(H) = \emptyset$ . The integer-magic spectra of certain classes of graphs resulted by the amalgamation of cycles and stars have already been identified [5], and in [6] the integer-magic spectra of the trees of diameter at most four have been completely characterized. Also, the integer-magic spectra of some other graphs have been studied in [4, 7, 8, 9, 11, 12, 13].

## 2. ZERO-SUM MAGIC GRAPHS

**Definition 2.1.** An  $h$ -magic graph  $G$  is said to be  $h$ -zero-sum (or just zero-sum) if there is a magic labeling of  $G$  in  $\mathbb{Z}_h$  that induces a vertex labeling with sum 0. The null set of a graph  $G$ , denoted by  $N(G)$ , is the set of all natural numbers  $h \in \mathbb{N}$  such that  $G$  is  $h$ -magic and admits a zero-sum labeling in  $\mathbb{Z}_h$ .

Clearly, a graph that has an edge pendant is not zero-sum. Also, the null set of a graph is contained in its integer-magic spectrum. The idea of the null set of a graph was introduced in [10] and the following results are established in that paper:

**Theorem 2.2.** If  $n \geq 4$ , then  $N(K_n) = \mathbb{N} - \{1 + (-1)^n\}$ .

**Theorem 2.3.** If  $m, n \geq 2$ , then  $N(K(m, n)) = \mathbb{N} - \{1 - (-1)^{m+n}\}$ .

**Null Sets of Cycle Related Graphs** There are different classes of cycle related graphs that have been studied for variety of labeling purposes. J. Gallian [2] has a nice collection of such graphs. Here are some results concerning the null sets of some of the cycle related graphs [10].

**Theorem 2.4.** If  $n \geq 3$ , then  $N(C_n) = \begin{cases} \mathbb{N} & \text{if } n \text{ is even;} \\ 2\mathbb{N} & \text{if } n \text{ is odd.} \end{cases}$

For any three positive integers  $\alpha < \beta \leq \gamma$ , the *theta graph*  $\theta_{\alpha,\beta,\gamma}$  consists of three edge disjoint paths of length  $\alpha, \beta$  and  $\gamma$  having the same endpoints, as illustrated in Figure 2. Theta graphs are also known as cycles with a  $P_k$  chord.

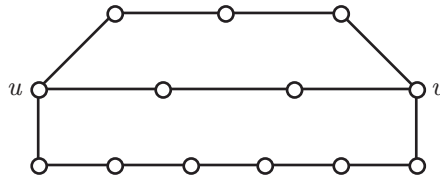


FIGURE 2. The graph  $\theta_{3,4,7}$ .

**Theorem 2.5.**  $N(\theta_{\alpha,\beta,\gamma}) = \begin{cases} \mathbb{N} - \{2\} & \text{if } \alpha, \beta, \gamma \text{ have the same parity;} \\ 2\mathbb{N} - \{2\} & \text{otherwise.} \end{cases}$

When  $k$  copies of  $C_n$  ( $n \geq 3$ ) share a common edge, it will form an  $n$ -gon book of  $k$  pages and is denoted by  $B(n, k)$ .

**Theorem 2.6.**  $N(B(n, k)) = \begin{cases} \mathbb{N} - \{1 + (-1)^k\} & \text{if } n \text{ is even;} \\ 2\mathbb{N} - \{1 + (-1)^k\} & \text{if } n \text{ is odd.} \end{cases}$

There are many other classes of cycle related graphs. *Wheels*  $W_n = C_n + K_1$  and *Fans* (also known as *Shells*) are among them. When  $n - 3$  chords in cycle  $C_n$  share a common vertex, the resulting graph is called *Fan (or Shell)* and is denoted by  $F_n$ , which is isomorphic to  $P_{n-1} + K_1$ . The integer-magic spectra of wheels and fans are determined in [9].

**Problem 2.7.** *Determine the null sets of wheels and fans.*

In magic labeling of a graph  $G$ , recognizing its components and the null sets of the components will be extremely helpful. For example, consider the graph  $G$  illustrated in Figure 3. This graph is constructed by six copies of  $K_4$ .

By theorem 2.2, the null set of  $K_4$  is  $\mathbb{N} - \{2\}$ . With this information and the fact that the applied construction preserves the zero-sum property, one can easily see that  $N(G) = IM(G) = \mathbb{N} - \{2\}$ .

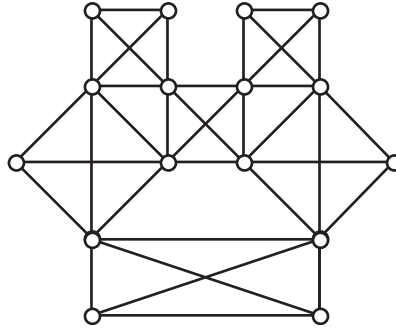


FIGURE 3. Find the integer-magic spectrum of this graph!!!

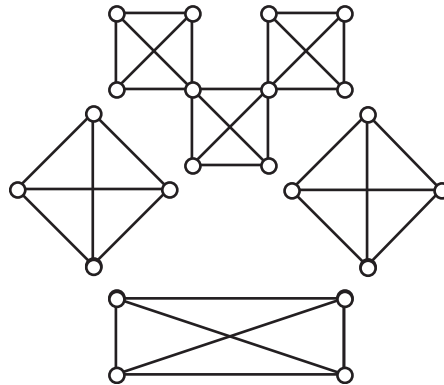


FIGURE 4. Graph  $G$  is constructed by six copies of  $K_4$ .

### 3. NULL SETS OF THE GENERALIZED THETA GRAPHS

Given  $k \geq 2$  the positive integers  $a_1 < a_2 \leq a_3 \leq \dots \leq a_k$ , the generalized theta graph  $\theta(a_1, a_2, \dots, a_k)$  consists of  $k$  edge disjoint paths of lengths  $a_1, a_2, \dots, a_k$  having the same initial and terminal points. In this section, the null sets of these graphs will be characterized. First one useful observation:

**Lemma 3.1. (Alternating label)** *Let  $v_1, v_2, v_3$  and  $v_4$  be four vertices of a graph  $G$  that are adjacent ( $v_1 \sim v_2 \sim v_3 \sim v_4$ ) and  $\deg v_2 = \deg v_3 = 2$ . Then in any magic labeling of  $G$  the edges  $u_1u_2$  and  $u_3u_4$  have the same label.*

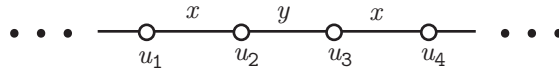


FIGURE 5. Alternating label in a magic labeling.

*Proof.* Let  $l : E(G) \rightarrow A - \{0\}$  be any magic labeling of  $G$  by the nonzero elements of an abelian group  $A$ . Then the condition  $l^+(u_2) = l^+(u_3)$  implies that  $l(u_1u_2) + l(u_2u_3) = l(u_2u_3) + l(u_3u_4)$  or  $l(u_1u_2) = l(u_3u_4)$ .  $\square$

When discussing magic labeling of a generalized theta graph  $G = \theta(a_1, a_2, \dots, a_k)$ , the alternating label lemma (3.1), allows us to assume that  $a_i = 2$  or  $3$ . For convenience, we will use  $\theta(2^m, 3^n)$  to denote the generalized theta graph which consists of  $m$  paths of even lengths and  $n$  paths of odd lengths. If the generalized theta graph consists of just paths of even (or odd) lengths, then we will use the notation  $\theta(2^m)$  (or  $\theta(3^n)$ ) and will require  $m > 1$  (or  $n > 1$ ). We observe that any magic labeling of  $\theta(2^m, 3)$  is similar to that of  $\theta(2^m, 1)$ , which is isomorphic to  $B(2p_i + 1, m)$ , the  $m$ -page book all of its pages are odd cycles. The null set of the latter is the same as that of  $B(3, m)$ , the 3-gon book of  $m$  pages. By 2.6, we have  $N(\theta(2^m, 3)) = N(B(3, m)) = 2\mathbb{N} - \{1 + (-1)^m\}$ . Similarly, by the alternating label lemma, any magic labeling of  $\theta(3^{n+1})$  is similar to that of  $\theta(3^n, 1)$ , which is isomorphic to  $B(2p_i, n)$ , the  $n$ -page book all of its pages are even cycles. The null set of the latter is the same as that of  $B(4, n)$ , the 4-gon book of  $n$  pages. By 2.6, we have  $N(\theta(3^{n+1})) = N(B(4, n)) = \mathbb{N} - \{1 + (-1)^n\}$ . Finally, we note that the magic labeling of  $\theta(2^m)$  is similar to that of  $K(2, m)$ . Therefore, by 2.3,  $N(\theta(2^m)) = N(K(2, m)) = \mathbb{N} - \{1 - (-1)^m\}$ .

**Lemma 3.2.** *For any positive integer  $n$ ,  $N(\theta(2, 3^n)) = 2\mathbb{N} - \{1 + (-1)^n\}$ .*

*Proof.* Note that if  $n = 1$ , then the graph  $\theta(2, 3^n)$  becomes an odd cycle and by 2.4, its null set is  $2\mathbb{N}$ , which is consistent with the statement of the lemma. Suppose  $n > 1$ . The condition  $l^+(u) \equiv l^+(v)$  implies that  $z + \sum x_i \equiv -z + \sum x_i$  or  $2z \equiv 0 \pmod{h}$ . Therefore,

$h$  has to be even. On the other hand, suppose  $h = 2r$  ( $r > 1$ ). We consider the following two cases:

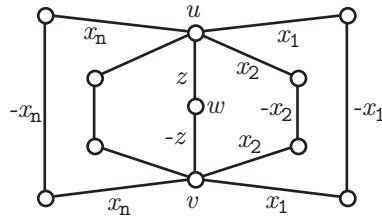


FIGURE 6. A potential magic labeling of  $\theta(2, 3^n)$  in  $\mathbb{Z}_h$ .

**Case I.** If  $n = 2p + 1$  is an odd number, then label all the edges by  $r$ . This will provide a magic labeling with sum 0.

**Case II.** If  $n = 2p$  is an even number, then let  $z = r$ ,  $x_1 = r - 1$  and  $x_i = (-1)^i$  for  $i \geq 2$ . This will provide a magic labeling with sum 0.

Finally, note that the degree set of this graph is  $\{2, n + 1\}$  and the graph is 2-magic if and only if  $n$  is odd. □

**Lemma 3.3.** For any two positive integers  $m, n > 1$ ,  $N(\theta(2^m, 3^n)) = \mathbb{N} - \{1 - (-1)^{m+n}\}$ .

*Proof.* Let  $u$  be one the endpoints of all the paths. We label the  $n$  edges of the paths of odd lengths incident with  $u$  by  $x_i$  ( $1 \leq i \leq n$ ) and the  $m$  edges of the paths of even lengths incident with  $u$  by  $y_i$  ( $1 \leq i \leq m$ ), as illustrated in Figure 7.

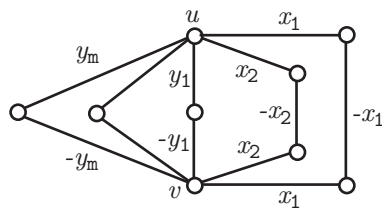


FIGURE 7. A potential magic labeling of  $\theta(2^m, 3^n)$ .

Note that  $\deg u = m + n$  and the graph is 2-magic if and only if  $m + n$  is even. Now, based on the parities of  $m$  and  $n$ , we will consider four different cases:

**Case 1.** If  $m$  and  $n$  are both even, we will label the edges adjacent to  $u$  by  $1, -1, \dots, 1, -1$ , as illustrated in Figure 8. This will provide a magic labeling with sum 0.

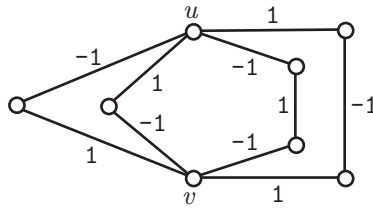


FIGURE 8.  $m$  and  $n$  are both even.

**Case 2.** If  $m$  is even and  $n$  is odd, then let  $x_1 = 2, x_2 = x_3 = -1, x_i = (-1)^i (3 < i \leq n)$  and  $y_j = (-1)^j (1 \leq j \leq m)$ . This will provide a magic labeling with sum 0.

**Case 3.** If  $m$  is odd and  $n$  is even, then let  $y_1 = 2, y_2 = y_3 = -1, y_i = (-1)^i (3 < i \leq m)$  and  $x_j = (-1)^j (1 \leq j \leq n)$ . This will provide a magic labeling with sum 0.

**Case 4.** If  $m$  and  $n$  are both odd, then let  $x_1 = y_1 = 2, x_2 = x_3 = y_2 = y_3 = -1, y_i = (-1)^i (3 < i \leq m)$  and  $x_j = (-1)^j (1 < j \leq n)$ . This will provide a magic labeling with sum 0. □

We conclude this section by the following theorem concerning the generalized theta graphs:

**Theorem 3.4.** *Following the above notations, for any two non-negative integers  $m, n$*

$$N(2^m, 3^n) = \begin{cases} 2\mathbb{N} - \{1 - (-1)^{m+n}\} & \text{if } m = 1 \text{ or } n = 1; \\ \mathbb{N} - \{1 - (-1)^{m+n}\} & \text{otherwise.} \end{cases}$$

#### 4. UNIFORMLY NULL GRAPHS

**Definition 4.1.** *An  $h$ -magic graph  $G$  is said to be uniformly null if every  $h$ -magic labeling of  $G$  induces 0 sum.*

**Theorem 4.2.** *The bipartite graph  $K(m, n)$  is uniformly null if and only if  $|m - n| = 1$ .*

*Proof.* Let  $S = \{u_1, u_2, \dots, u_m\}$  and  $T = \{v_1, v_2, \dots, v_m, v_{m+1}\}$  be the two partite sets of  $K(m, m + 1)$  and let  $l : K(m, m + 1) \rightarrow \mathbb{Z}_h$  be any magic labeling of this graph. From  $l^+(u_i) = l^+(v_i)$  we get

$$(4.1) \quad \sum_{i=1}^m l^+(u_i) = \sum_{i=1}^m l^+(v_i).$$

The left hand side of (4.1) is the sum of the labels of all edges of the graph, while the right hand side is the sum of labels of all edges except those that are incident with  $v_{m+1}$ . Therefore, (4.1) implies that  $l^+(v_{m+1}) = 0$  and the graph is uniformly null.

Now suppose  $|m - n| \neq 1$ , without loss of generality we may assume that  $m > n + 1$ . To show that  $K(m, n)$  is not uniformly null, we will present an  $h \in \mathbb{N}$  and a labeling of  $K(m, n)$  in  $\mathbb{Z}_h$  with nonzero sum. Consider the following two cases:

**Case I.** Suppose  $(m - n) \nmid n$ . Choose  $h = m - n$  and label all the edges by 1. This provides a magic labeling of  $K(m, n)$  with sum  $n$ .

**Case II.** Suppose  $(m - n) | n$ . Choose  $h = m - n$  and label the edges by

$$l(u_i v_j) = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}$$

as demonstrated in the following table (4.2):

	$v_1$	$v_2$	$v_3$	$\dots$	$v_n$	$v_{n+1}$	$\dots$	$v_m$
$u_1$	2	1	1	$\dots$	1	1	$\dots$	1
$u_2$	1	2	1	$\dots$	1	1	$\dots$	1
$u_3$	1	1	2	$\dots$	1	1	$\dots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
$u_n$	1	1	1	$\dots$	2	1	$\dots$	1

This provides a magic labeling of  $K(m, n)$  with sum 1. □

We conclude this paper by the following problem:

**Problem 4.3.** *In 4.2 it was shown that for any  $n \geq 2$ , the complete bipartite graph  $K(n, n+1)$  is uniformly null. Identify another class of graphs whose elements are uniformly null.*

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