ZERO-SUM MAGIC GRAPHS AND THEIR NULL SETS

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Abstract. For any $h \in \mathbb{N}$, a graph $G = (V, E)$ is said to be h-magic if there exists a labeling $l : E(G) \to \mathbb{Z}_h - \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \to \mathbb{Z}_h$ defined by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. When this constant is 0 we call G a zero-sum h-magic graph. The *null set* of G is the set of all natural numbers $h \in \mathbb{N}$ for which G admits a zero-sum h-magic labeling. In this paper we will identify several classes of zero sum magic graphs and will determine their null sets.

Key Words: magic, non-magic, zero-sum, null set.

AMS Subject Classification: 05C15 05C78

1. Introduction

For an abelian group A, written additively, any mapping $l : E(G) \to A - \{0\}$ is called a *labeling*. Given a labeling on the edge set of G one can introduce a vertex set labeling $l^+ : V(G) \to A$ by

$$l^+(v) = \sum_{uv \in E(G)} l(uv).$$

A graph G is said to be A-*magic* if there is a labeling $l : E(G) \to A - \{0\}$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; that is, $l^+(v) = c$ for some fixed $c \in A$. In general, a graph G may admit more than one labeling to become A-magic; for example, if $|A| > 2$ and $l : E(G) \to A - \{0\}$ is a magic labeling of G with sum c, then $\lambda : E(G) \to A - \{0\}$, the *inverse labeling* of l, defined by $\lambda(uv) = -l(uv)$ will provide another magic labeling of G with sum $-c$. A graph $G = (V, E)$ is called *fully magic* if it

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is A-magic for every abelian group A. For example, every regular graph is fully magic. A graph $G = (V, E)$ is called *non-magic* if for every abelian group A, the graph is not A-magic. The most obvious example of a non-magic graph is P_n $(n \geq 3)$, the path of order n. As a result, any graph with a path pendant of length $n \geq 3$ would be non-magic. Here is another example of a non-magic graph: Consider the graph H Figure 1. Given any abelian group A, a typical magic labeling of H is illustrated in that figure. Since $l^{+}(u) = x \neq 0$ and $l^{+}(v) = 0$, H is not A-magic. This fact can be generalized as follows:

Lemma 1.1. *Every even cycle C_n with $2k + 1\,(< n)$ consecutive pendants is non-magic.*

Lemma 1.2. *Every odd cycle C_n with $2k\,(< n)$ consecutive pendants is non-magic.*

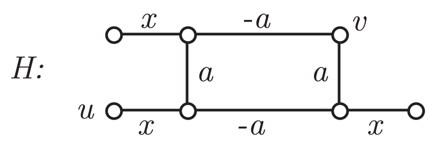

FIGURE 1. An example of non-magic graph.

Certain classes of non-magic graphs are presented in [1].

The original concept of A-magic graph is due to J. Sedlacek [11, 12], who defined it to be a graph with a real-valued edge labeling such that

(1) distinct edges have distinct nonnegative labels; and

(2) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Jenzy and Trenkler [4] proved that a graph G is magic if and only if every edge of G is contained in a (1-2)-factor. $\mathbb{Z}$-magic graphs were considered by Stanley [13, 14], who pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. Recently, there has been considerable research articles in graph labeling, interested readers are directed to [3, 15]. For convenience, the notation 1-magic will be used to indicate $\mathbb{Z}$-magic and $\mathbb{Z}_h$-magic graphs will be referred to as h-magic graphs. Clearly, if a graph is h-magic, it is not necessarily k-magic $(h \neq k)$.

Definition 1.3. *For a given graph G the set of all positive integers h for which G is h-magic is called the* integer-magic spectrum *of G and is denoted by $IM(G)$.*

Since any regular graph is fully magic, then it is h-magic for all positive integers $h \geq 2$; therefore, $IM(G) = I\!N$. On the other hand, the graph H, Figure 1, is non-magic, hence $IM(H) = \emptyset$. The integer-magic spectra of certain classes of graphs resulted by the amalgamation of cycles and stars have already been identified [5], and in [6] the integer-magic spectra of the trees of diameter at most four have been completely characterized. Also, the integer-magic spectra of some other graphs have been studied in [7, 8, 9, 10].

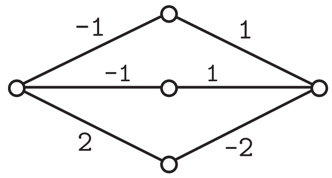

FIGURE 2. The graph $K(2,3)$ is h-magic ($\forall h \geq 3$).

Definition 1.4. *An h-magic graph G is said to be h-zero-sum (or just zero-sum) if there is a magic labeling of G in $\mathbb{Z}_h$ that induces a vertex labeling with sum 0. The graph G is said to be uniformly null if every magic labeling of G induces 0 sum.*

Clearly, a graph that has an edge pendant is not zero-sum. Here is an example of a uniformly null graph:

Lemma 1.5. *The complete bipartite graph $K(2,3)$ is a uniformly null magic graph.*

Proof. Since the degree set of $K(2,3)$ is $\{2,3\}$, it is not 2-magic. On the other hand, the labeling presented in Figure 2 indicates that the integer-magic spectrum of $K(2,3)$ is $I\!N - \{2\}$ with sum being 0. Now we wish to show that 0 is the only possible sum. Consider an arbitrary labeling of $K(2,3)$, as illustrated in Figure 3.

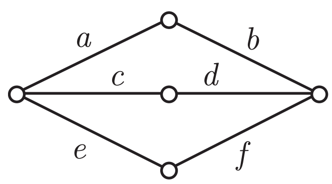

FIGURE 3. An arbitrary labeling of $K(2,3)$.

For $K(2,3)$ to be h-magic, we require that

$$\begin{aligned} a+c+e &\equiv a+b \pmod{h}; \\ b+d+f &\equiv c+d \pmod{h}. \end{aligned}$$

If we add these equations, we get $e+f=0$. Hence, the induced sum cannot be nonzero. $\square$

Definition 1.6. *The null set of a graph G, denoted by $N(G)$, is the set of all natural numbers $h \in I\!N$ such that G is h-magic and admits a zero-sum labeling in $\mathbb{Z}_h$.*

One can introduce a number of operations among zero-sum graphs which produce magic graphs. Frucht and Harary [2] introduced the *corona* of two G and H, denoted by $G©H$, to be the graph with base G such that each vertex $v \in V(G)$ is joined to all vertices of a separate copy of H.

Observation 1.7. *If G has zero-sum in $\mathbb{Z}_h$, then $G©K_1$ is h-magic.*

A graph G with a fixed vertex $u \in V(G)$ will be denoted by the order pair (G,u). Given two ordered pair (G,u) and (H,v), one can construct another graph by linking these two graphs through identifying the vertices u and v. We will use the notation $(G,u) \diamond (H,v)$ for this construction or simply $G \diamond H$ if there is no ambiguity about the choices of u and v.

Definition 1.8. *Given n graphs G_i $(i = 1, 2 \cdots, n)$, the chain $G_1 \diamond G_2 \diamond \cdots \diamond G_n$ is the graph in which one of the vertices of G_i is identified with one of the vertices of G_{i+1}. If $G_i = G$, we use the notation $\diamond G^n$ for the n-link chain all of whose links are G.*

Observation 1.9. *If graphs G_i have zero sum, so does the chain $G_1 \diamond G_2 \diamond \cdots \diamond G_n$, hence it is a magic graph. Moreover, if $G_i = G$, then the null set of the chain $\diamond G^n$ is the same as $N(G)$.*

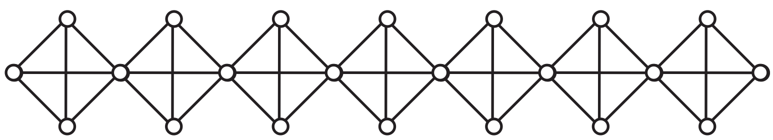

FIGURE 4. A 7-link chain whose links are K_4

With the notation in 1.8, if we further identify one of the vertices of G_n by another vertex of G_1, the resulting graph is a necklace. Similarly, all the bids of this necklace can be the same graph G, for which we have the following observation:

Observation 1.10. *If the graphs G_i have zero sum, so does the necklace formed by these graphs. Moreover, if $G_i = G$, then the null set of this necklace is the same as $N(G)$.*

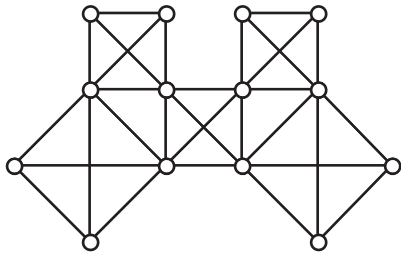

FIGURE 5. Find the integer-magic spectrum of this graph!!!

These are just a few operations among the graphs that preserve the magic property, when the graphs are zero-sum. In magic labeling of graphs, knowing the components of a graph and the null sets of the components will be extremely helpful. For example, consider the graph G illustrated in Figure 5. This graph is constructed by five copies of K_4. In the next section, (theorem 2.1), it is shown that the null set of K_4 is $I\!N - \{2\}$. With this information and the fact that the applied construction preserves the zero-sum property, one can easily see that $N(G) = IM(G) = I\!N - \{2\}$. In the following sections the null sets of a few well known classes of graphs will be characterized.

2. Null Sets of Complete Graphs

Complete graphs being regular are fully magic, hence their integer-magic spectrum is $I\!N$. In this section we will determine the null sets of these graphs. Note that $K_3 \equiv C_3$ and $N(K_3) = 2I\!N$. In what follows we will assume that $n \geq 4$.

Theorem 2.1. *If $n \geq 4$, then* $N(K_n) = \begin{cases} I\!N & \text{if } n \text{ is odd;} \\ I\!N - \{2\} & \text{if } n \text{ is even.} \end{cases}$

Proof. Let $u_1,\ u_2,\ \cdots, u_n$ be the vertices of K_n and assume that they are arranged counterclockwise around a circle. If n is even, then $\deg(u_i)$ is odd and K_n cannot have zero sum in $\mathbb{Z}_2$. Also, with the following convention, we will use u_j as one of the vertices even if $j \neq 1, 2, \cdots, n$: Let $u_j = \begin{cases} u_{j-n} & \text{if } j > n \text{ ;} \\ u_{j+n} & \text{if } j \leq 0. \end{cases}$ To prove the theorem, we will consider the following five cases, in each case we will introduce an appropriate labeling $l : E(K_n) \to \mathbb{Z}_3$ with sum 0.

Case 1. n is odd and $n = 4p + 1$. In this case, labeling of the edges are done by

$$l(u_iu_j) = \begin{cases} 1 & \text{if } \ j = i \pm r \ (1 \leq r \leq p); \\ -1 & \text{otherwise.} \end{cases}$$

Since the $\deg(u_i) = n - 1 = 4p$, there are $4p$ edges that are incident with vertex u_i, half of which are labeled 1 and the other half -1. Therefore, $l^+(u_i) = 0$ for all $i = 1, 2, , \cdots, n$.

Case 2. n is odd and $n = 4p + 3$. In this case, we label the edges by

$$l(u_iu_j) = \begin{cases} 2 & \text{if } \ j = i + 1; \\ 1 & \text{if } \ j = i \pm r \ (2 \leq r \leq p); \\ -1 & \text{otherwise.} \end{cases}$$

Since the $\deg(u_i) = n - 1 = 4p + 2$, there are $4p + 2$ edges incident with this vertex, two edges are labeled 2, $2p - 2$ edges are labeled 1, and the remaining $2p + 2$ edges are labeled -1. Therefore, $l^+(u_i) = 0$ for all $i = 1, 2, , \cdots, n$. This labeling is illustrated in table (2.1).

(2.1)

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	$*$	2	-1	-1	-1	-1	2
u_2	2	$*$	2	-1	-1	-1	-1
u_3	-1	2	$*$	2	-1	-1	-1
u_4	-1	-1	2	$*$	2	-1	-1
u_5	-1	-1	-1	2	$*$	2	-1
u_6	-1	-1	-1	-1	2	$*$	2
u_7	2	-1	-1	-1	-1	2	$*$

Case 3. n is even and $n = 6p + 4$. In this case, we label the edges by

$$l(u_iu_j) = \begin{cases} 2 & \text{if } \ j - i = 3p + 2; \\ 2 & \text{if } \ j = i \pm r \ (1 \leq r \leq p); \\ -1 & \text{otherwise.} \end{cases}$$

Note that u_iu_j $(j - i = 3p + 2)$ are the opposite vertices, u_iu_j $(j = i + r)$ are on the left of u_i, and u_iu_j $(j = i - r)$ are on the right of u_i. Since the $\deg(u_i) = n - 1 = 6p + 3$, there are $6p + 3$ edges incident with this vertex. We label $2p + 1$ of them by 2 (opposite, p on the left, and p on the right) and the remaining $4p + 2$ by -1. Therefore, $l^+(u_i) = 0$ for all $i = 1, 2, , \cdots, n$.

Case 4. n is even and $n = 6p + 2$. In this case, we label the edges by

$$l(u_iu_j) = \begin{cases} 2 & \text{if } \ j - i = 3p + 2; \\ 2 & \text{if } \ j = i \pm r \ (2 \leq r \leq p); \\ 1 & \text{if } \ j = i \pm 1; \\ -1 & \text{otherwise.} \end{cases}$$

Since the $\deg(u_i) = n - 1 = 6p + 1$, there are $6p + 1$ edges incident with this vertex. We label $2p-1$ of them by 2 (opposite, $p-1$ on the left, and $p-1$ on the right), two edges by 1 (immediate left and right), and the remaining $4p$ by -1. Therefore, $l^+(u_i) = 0$ for all $i = 1, 2, , \cdots, n$.

Case 5. n is even and $n = 6p$. In this case, we label the edges by

$$l(u_iu_j) = \begin{cases} 2 & \text{if } j - i = 3p + 2; \\ 2 & \text{if } j = i \pm r \ (2 \le r \le p); \\ 1 & \text{if } j = i + 1 \ (i = 2r - 1); \\ -1 & \text{otherwise.} \end{cases}$$

Since the $\deg(u_i) = n - 1 = 6p - 1$, there are $6p - 1$ edges incident with this vertex. We label $2p - 1$ of them by 2 (opposite, $p - 1$ on the left, and $p - 1$ on the right), one edges by 1, and the remaining $4p$ by -1. Therefore, $l^+(u_i) = 0$ for all $i = 1, 2, , \cdots, n$. This labeling is illustrated in table (2.2).

(2.2)

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$*$	1	-1	2	-1	-1
u_2	1	$*$	-1	-1	2	-1
u_3	-1	-1	$*$	1	-1	2
u_4	2	-1	1	$*$	-1	2
u_5	-1	2	-1	-1	$*$	1
u_6	-1	-1	2	-1	1	$*$

□

3. Null Sets of Complete Bipartite Graphs

Theorem 3.1. *Let* $m, n \ge 2$. *Then*

$$N(K(m,n)) = \begin{cases} I\!N & \text{if } m + n \text{ is even;} \\ I\!N - \{2\} & \text{if } m + n \text{ is odd.} \end{cases}$$

Proof. Let $S = \{u_1, u_2, \cdots, u_m\}$ and $T = \{v_1, v_2, \cdots, v_n\}$ be the two partite sets. In labeling of edges $u_i\, v_j$, with elements of $Z\!\!Z_h$ $(h \ge 3)$, we will consider three cases:

Case I. $m,\ n$ are both even. We label the edges by $l(u_iv_j) = (-1)^{i+j}$. This will result in $l^+ \equiv 0$.

Case II. m is even and n is odd. We label the edges by

$$l(u_iv_j) = \begin{cases} 2 \cdot (-1)^{i-1} & \text{if } j = 1 \\ (-1)^i & \text{if } j = 2, 3 \\ (-1)^{i+j} & \text{otherwise} \end{cases}$$

This labeling is illustrated in table (3.1).

$$
\begin{array}{l|rrr|rcr}
 & v_1 & v_2 & v_3 & v_4 & \dots & v_n \\
\hline
u_1 & 2 & -1 & -1 & -1 & \dots & 1 \\
u_2 & -2 & 1 & 1 & 1 & \dots & -1 \\
u_3 & 2 & -1 & -1 & -1 & \dots & 1 \\
u_4 & -2 & 1 & 1 & 1 & \dots & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_{m-1} & 2 & -1 & -1 & -1 & \dots & 1 \\
u_m & -2 & 1 & 1 & 1 & \dots & -1
\end{array}
\tag{3.1}
$$

Case III. $m,\ n$ are both odd. We label the edges by using the following table (8):

$$
\begin{array}{l|rrr|rrccr}
 & v_1 & v_2 & v_3 & v_4 & v_5 & & \dots & v_n \\
\hline
u_1 & 2 & -1 & -1 & 2 & -2 & & \dots & -2 \\
u_2 & -1 & 2 & -1 & -1 & 1 & & \dots & 1 \\
u_3 & -1 & -1 & 2 & -1 & 1 & & \dots & 1 \\
\hline
u_4 & 2 & -1 & -1 & \ddots & & & & \\
u_5 & -2 & 1 & 1 & & & & & \\
\vdots & \vdots & \vdots & \vdots & & & (-1)^{i+j} & & \\
u_m & 2 & -1 & -1 & & & & & \ddots
\end{array}
\tag{3.2}
$$

Finally, we observe that if m, n have different parity, the graph would not be 2-magic. $\square$

4. Null Sets of Cycle Related Graphs

There are different classes of cycle related graphs that have been studied for variety of labeling purposes. J. Gallian [3] has a nice collection of such graphs. In this section, the null sets of some of the cycle related graphs are investigated. First, one useful observation:

Observation 4.1. *In any magic labeling of of a cycle the edges should alternatively be labeled the same elements of the group.*

Proof. Let $u_1,\ u_2,\ u_3$, and u_4 be the four consecutive vertices of a cycle. The requirement of $l(u_1u_2) + l(u_2u_3) = l(u_2u_3) + l(u_3u_4)$ implies that $l(u_1u_2) = l(u_3u_4)$. $\square$

Since a cycle is a 2-regular graph, it is fully magic. Therefore, its integer-magic spectrum is $I\!N$. For the null-set of C_n we have the following theorem:

Theorem 4.2. $N(C_n) = \begin{cases} I\!N & \text{if } n \text{ is even;} \\ 2I\!N & \text{if } n \text{ is odd .} \end{cases}$

Proof. If n is even, then there are even number of edges and we label every other edge by 1 and -1. If n is odd, then in any magic labeling of C_n, all the edges are labeled the same element of $x \in \mathbb{Z}_h$. As a result, for C_{2k+1} to be zero-sum, one needs $2x \equiv 0 \pmod h$ or $2|h$. On the other hand, if $h = 2r$, then the choice of $x = r$ will result to the zero-sum magic labeling of C_{2k+1}. $\square$

A *cycles with a P_k chord* is a cycle with the path P_k joining two nonconsecutive vertices of the cycle. Since the degree set of these graphs is $\{2, 3\}$, they are not 2-magic. Based on Observation 4.1, it is enough to consider the cases when $k = 2, 3$. The chord P_k splits C_n into two subcycles. Depending on the number of edges of these subcycles, we will have different results for the null set. The next lemma is about cycles with a P_2 chord:

Lemma 4.3. *Let $G_{n,2}$ be the cycle C_n with a P_2 chord. Then*
$$N(G_{n,2}) = \begin{cases} \mathbb{N} - \{2\} & \text{both subcycles are even;} \\ 2\mathbb{N} - \{2\} & \text{otherwise .} \end{cases}$$

Proof. Since the degree set of $G_{n,2}$ is $\{2, 3\}$, the graph is not 2-magic. Now based on the observation 4.1, it is enough to consider C_3 and C_4 as the two subcycles.

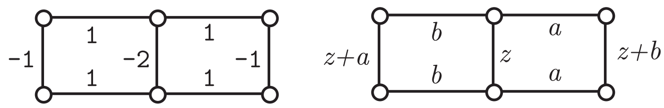

FIGURE 6. $G_{n,2}$ consists of two even subcycles.

Case I. Both subcycles are even. The labeling illustrated in Figure 6, proves that the integer-magic spectrum of $G_{n,2}$ is the same as its null set; that is, $N(G_{n,2}) = IM(G_{n,2}) = \mathbb{N} - \{2\}$.

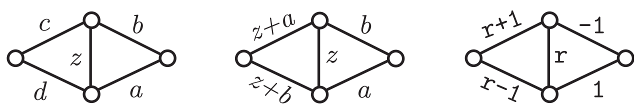

FIGURE 7. $G_{n,2}$ consists of two odd subcycles.

Case II. Both subcycles are odd. The typical labeling of $G_{n,2}$ in $\mathbb{Z}_h$ is illustrated in Figure 7. The requirement $a + z + d = c + d$ and $b + z + c = c + d$ imply that $c = a + z$ and $d = b + z$. Also, $a + b = c + d$ will result to $2z \equiv 0 \pmod h$ or $2|h$. On the other hand, if $h = 2r$, then the choice of $z = r$, $a = 1$, and $b = -1$ provides a zero sum result. Therefore, $IM(G_{n,2}) = N(G_{n,2}) = 2\mathbb{N} - \{2\}$.

Case III. Subcycles have different parities. The typical labeling of $G_{n,2}$ in $\mathbb{Z}_h$ is illustrated in Figure 8. The condition $a+x+z = a+y+z$ implies $x = y$. Also, the requirements $a+z+x = 2x$ will result to $z = x - a$ and $b = 2x - a$. Therefore, given $x \in \mathbb{Z}_h - \{0\}$, we need another nonzero element $a \neq x$, $2x$, hence $h \geq 4$. Therefore, the integer-magic spectrum of such graphs would be $\mathbb{N} - \{2, 3\}$, while the null set is $2\mathbb{N} - \{2\}$. □

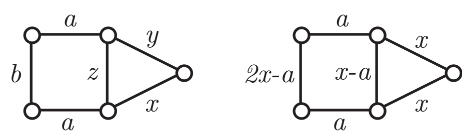

FIGURE 8. $G_{n,2}$ consists of one odd and one even subcycles.

Corollary 4.4. *C_n with a P_2 chord is not uniformly null.*

Lemma 4.5. *Let $G_{n,3}$ be the cycle C_n with a P_3 chord. Then*
$$N(G_{n,3}) = \begin{cases} \mathbb{N} - \{2\} & \textit{both subcycles are even;} \\ 2\mathbb{N} - \{2\} & \textit{otherwise .} \end{cases}$$

Proof. Based on the observation 4.1, it is enough to consider C_4 and C_5 as the two subcycles.

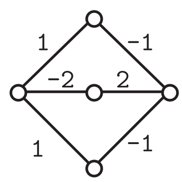

FIGURE 9. $G_{n,3}$ consists of two even subcycles.

Case I. Both subcycles are C_4. The labeling illustrated in Figure 9, shows that the integer-magic spectrum of $G_{n,3}$ is the same as its null set; that is, $N(G_{n,3}) = IM(G_{n,3}) = \mathbb{N} - \{2\}$.

Case II. Both subcycles are C_5. The typical magic labeling of $G_{n,3}$ in $\mathbb{Z}_h$ is illustrated in Figure 10, which has sum $2x$. Here, given $x \in \mathbb{Z}_h$, one needs another nonzero element $a \neq x, -x$. Hence, the graph cannot be 3-magic, and its integer-magic spectrum is $\mathbb{N} - \{2, 3\}$. However, for the graph to have zero sum, we need $2x \equiv 0 \pmod{h}$; that is, h has to be even. Therefore, its null set in contained in $2\mathbb{N} - \{2\}$. On the other hand, if $h = 2r$, then the choices of $x = r$ and $a = 1$ provide a zero sum result. Therefore, $N(G_{n,3}) = 2\mathbb{N} - \{2\}$.

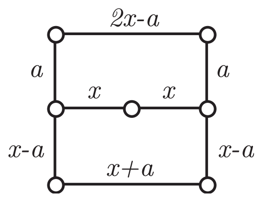

FIGURE 10. $G_{n,3}$ consists of two odd subcycles.

Case III. Subcycles have different parities. The typical magic labeling of $G_{n,3}$ in $\mathbb{Z}_h$ is illustrated in Figure 8. For the graph to be magic, we need $3a+c+x = a+c+x$ or $2a \equiv 0 \pmod{h}$; that is, h is even and the integer-magic spectrum of the graph would be $2\mathbb{N} - \{2\}$. For the graph to have zero sum, we need the additional condition $a+c+x \equiv 0 \pmod{h}$, that is always possible. One such labeling has been provided in Figure 11. Thus $IM(G_{n,3}) = N(G_{n,3}) = 2\mathbb{N} - \{2\}$. □

Corollary 4.6. *C_n with a P_3 chord is not uniformly null.*

We summarize the above 4.3 and 4.5 in the following theorem:

Theorem 4.7. *Let $G_{n,k}$ be the cycle C_n with a P_k chord. Then*
$$N(G_{n,k}) = \begin{cases} \mathbb{N} - \{2\} & \textit{both subcycles are even;} \\ 2\mathbb{N} - \{2\} & \textit{otherwise .} \end{cases}$$
Moreover, $G_{n,k}$ is not a uniformly null graph.

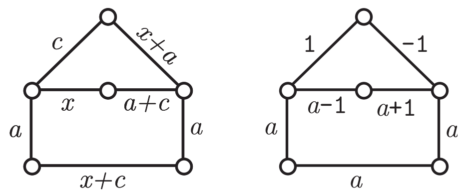

FIGURE 11. $G_{n,3}$ consists of one odd and one even subcycles.

When k copies of C_n share a common edge, it will form an *n-gon book of k pages* and is denoted by $B(n,k)$.

Theorem 4.8. $N(B(n,k)) = \begin{cases} \mathbb{N} & n \text{ is even, } k \text{ is odd;} \\ \mathbb{N} - \{2\} & n \text{ and } k \text{ are both even;} \\ 2\mathbb{N} - \{2\} & n \text{ is odd, } k \text{ is even;} \\ 2\mathbb{N} & n \text{ and } k \text{ are both odd.} \end{cases}$

Proof. Depending on whether n is even or odd it will be enough to consider C_4 and C_3, respectively. If n is even and k is odd, we will label the common edge by -1 and top edges $1,\ -1$ alternatively. This is a zero sum magic labeling.

If n and k are both even, we will label the common edge by -1 and one top edge by 2 the remaining top edges $-1,\ 1$ alternatively. This provides a zero sum magic labeling. Note that, in this case, the degrees of vertices do not have the same parity and the book is not 2-magic.

Suppose n is odd (C_3). We label the common edge by z and the i^{th} cycle edges by $a_i,\ -a_i$ as illustrated in Figure 12.

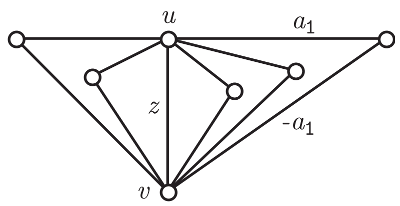

FIGURE 12. A typical zero sum magic labeling of $B(3, k)$.

The requirements $l^+(u) = l^+(v) = 0$ will lead us to the equations $z + \sum a_i = z - \sum a_i = 0$ or $2z \equiv 0 \pmod{h}$, which implies that h is even ($z \neq 0$). On the other hand if $h = 2r$ is even, then we consider two cases:

case I. If k is odd, we label all the edges by r which results in a zero sum magic labeling.

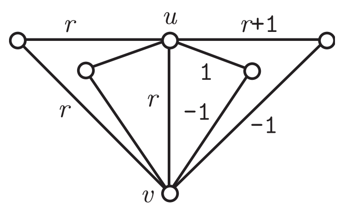

FIGURE 13. A zero sum magic labeling of $B(3, 2k)$.

case II. If k is even, we choose $a_1 = r - 1,\ a_2 = 1$, and $z = a_i = r$ ($i \geq 3$), as illustrated in Figure 13.

Finally, we observe that when n is odd and k is even, the book cannot be 2-magic. Therefore, the null space would be $2I\!N - \{2\}$. $\square$

There are many other classes of cycle related graphs. *Wheels* $W_n = C_n + K_1$ and Fans (also known as Shells) are among them. When $n-3$ chords in cycle C_n share a common vertex, the resulting graph is called *Fan (or Shell)* and is denoted by F_n, which is isomorphic to $P_{n-1} + K_1$. We conclude this paper by the following problems:

Problem 4.9. *Find the null sets of W_n and F_n.*

Problem 4.10. *In 1.5 it was shown that $K(2,3)$ is a uniformly null graph. Identify a class of graphs whose elements are uniformly null.*

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