NON-MAGIC AND K-NONMAGIC GRAPHS

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Abstract. Given an abelian group A, a graph G is said to be A-magic if there exists a labeling $l : E(G) \to A - \{0\}$ such that the induced vertex labeling $l^+ : V(G) \to A$ defined by

$$l^+(v) = \sum_{u \in N(v)} l(uv)$$

is a constant map. A graph G is said to be non-magic if for any abelian group A, it is not A-magic. Also, a $\mathbb{Z}$-magic graph G is said to be K-nonmagic if G is not $\mathbb{Z}_h$-magic for all $h = 1, 2, 3, \cdots, K$. In this paper, we will introduce a few classes of non-magic graphs. Then we will focus on the groups $\mathbb{Z}_h$ $(h \in \mathbb{N})$ and will investigate certain K-nonmagic graphs.

Key Words: magic, non-magic, integer-magic spectrum.

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1. Introduction

For any abelian group A, written additively, any mapping $l : E(G) \to A - \{0\}$ is called a *labeling*. Given a labeling l, one can introduce a vertex labeling $l^+ : V(G) \to A$ by

$$l^+(v) = \sum_{u \in N(v)} l(uv),$$

where $N(v)$ denotes the set of all vertices of G that are adjacent with v. A graph G is said to be *A-magic* if there is a labeling $l : E(G) \to A - \{0\}$ such that $l^+(v) = c$ $(v \in V(G))$ for some fixed $c \in A$. In general, a graph G may admit more than one magic-labeling; for example, if $|A| > 2$ and $l : E(G) \to A - \{0\}$ is a magic labeling of G with sum c, then $\lambda : E(G) \to A - \{0\}$, the inverse labeling of l, defined by $\lambda(uv) = -l(uv)$ will provide another magic labeling of G with sum $-c$.

The original concept of A-magic graph is due to J. Sedlacek [17, 18], who defined it to be a graph with a real-valued edge labeling such that

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(1) distinct edges have distinct nonnegative labels; and

(2) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

When $A = \mathbb{Z}$, the $\mathbb{Z}$-magic graphs were considered in Stanley [19, 20], he pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. When the group is $\mathbb{Z}_h$, we shall refer to the $\mathbb{Z}_h$-magic graph as *h-magic.* For a given graph G the set of all positive integers h for which G is $\mathbb{Z}_h$-magic (or simply h-magic) is called the *integer-magic spectrum* of G and is denoted by $IM(G)$. For more general results on integer-magic spectra of graphs, the reader is referred to [12, 13, 14]. In fact, in recent years, there has been great research interest in edge and vertex labeling of graphs with applications in different areas. For further reading, citations of a number of articles have been provided in the bibliography.

2. Non-magic Graphs

A graph $G = (V, E)$ is called *non-magic* if for every abelian group A it is not A-magic. The simplest non-magic graph is P_3, the path of order three.

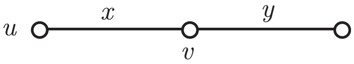

Figure 1. P_3, the path of order three, is non-magic.

Here, we need to have $l^+(u) = l^+(v)$ or $x = x + y$, which implies $y = 0$. In fact, For any $n \geq 3$, the path of order n is non-magic.

Observation 2.1. *Any graph with a pendant path of length at least two is non-magic.*

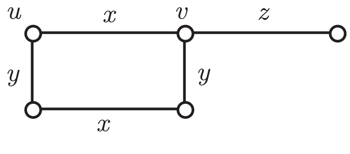

Figure 2. The cycle C_4 with an edge pendant is non-magic.

Observation 2.2. C_4, *the cycle of order four, with an edge pendant is non-magic. In general, any even cycle* C_{2n} *with an edge pendant is non-magic.*

Proof. As illustrated in the Figure 2, in any labeling, the sum of labels of the edges incident with vertex v needs to be equal to the sum of labels of the edges incident with u; that is, $x+y+z = x+y \Longrightarrow z=0$, a contradiction. $\square$

In this paper we will use $\mathbb{N}$ to denote the set of positive integers and

$$k+\mathbb{N} = \{\ k+n\ :\ n \in \mathbb{N}\ \} \ \text{ and } \ k\mathbb{N} = \{\ kn\ :\ n \in \mathbb{N}\ \}.$$

When k copies of C_n share a common edge it will form the n-gon book of k pages and is denoted by $B(n,k)$.

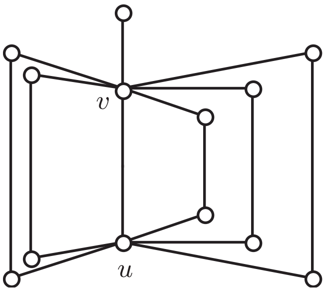

FIGURE 3. $B(4,5)$, the 4-gon book of five pages with common edge uv.

Theorem 2.3. *For any $k,n \in \mathbb{N}$, when we attach a pendant edge at one of the vertices of the common edge of $B(2n,k)$, the resulting graph will be non-magic.*

Another class of non-magic graphs is provided by certain trees of diameter 3. In general, trees of diameter three are called *double-stars*. These graphs have two central vertices u and v plus leaves. We will use $DS(m,n)$ to denote the double-star whose two central vertices have degrees m and n, respectively.

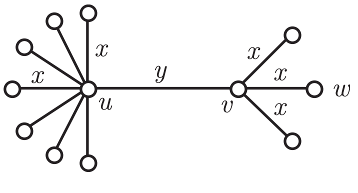

FIGURE 4. A typical magic labeling of $DS(m,n)$.

Theorem 2.4. *Double-star $DS(n+r,\, n)$ is non-magic if and only if $r|(n-2)$.*

Proof. Let l be any labeling of $DS(n+r,\, n)$. Then the condition $l^+(u) = l^+(v)$ gives us

$$(n+r-1)x + y = (n-1)x + y$$

or

$$rx = 0. \tag{2.1}$$

And condition $l^+(v) = x$ implies that

$$(n-2)x + y = 0. \tag{2.2}$$

If $r|(n-2)$, since $rx = 0$, then $y = 0$, which is not an acceptable magic labeling. Therefore, $DS(n+r,\, n)$ is non magic.

On the other hand, if r does not divide $n-2$, then there is a prime number p and a non-negative integer α such that $p^\alpha|(n-2)$, $p^{\alpha+1}|r$, but $p^{\alpha+1}$ does not divide $n-2$. Now, $x = 1$ and $y = 2-n$ are nonzero elements of $\mathbb{Z}_h$ $(h = p^{\alpha+1})$ that provide a magic labeling for $DS(n+r,\, n)$. Therefore, $DS(n+r,\, n)$ is h-magic. □

Examples 2.5.

(a) The double-star $DS(6,4)$ is non-magic; here $r = m - n = 2$ is a divisor of $n-2 = 2$.

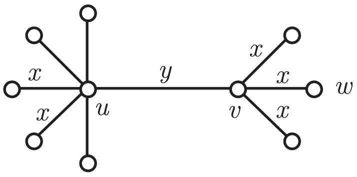

Figure 5. Double-star $DS(6,4)$ is non-magic.

(b) The double-star $DS(11,8)$ is non-magic; here $r = m - n = 3$ is a divisor of $n-2 = 6$.
(c) The double-star $DS(25,24)$ is non-magic; here $r = m - n = 1$.
(d) The double-star $DS(m,m)$ is non-magic if and only if $m = 2$.

Definition 2.6. A tree of diameter four, denoted by $TF(n; a_1, a_2, \cdots, a_n)$, consists of n stars $ST(a_1), ST(a_2), \cdots, ST(a_n)$ one of their edges is incident with a common vertex and $a_i \geq 2$ for at least two values of i. The common vertex will be called the center of the tree and will be denoted by c. Equivalently, $TF(n; a_1, a_2, \cdots, a_n)$ is a tree with center-vertex c, in which n edges $\{cu_1, cu_2, \cdots, cu_n\}$ are emanated from c, and $\deg(u_i) = a_i$ for each $i = 1, 2, \cdots, n$, as illustrated in the Figure 6.

In order to have a tree of diameter four, one needs $n \geq 2$ and $a_i \geq 2$ for at least two values of i.

Trees of diameter four are another source of non-magic graphs:

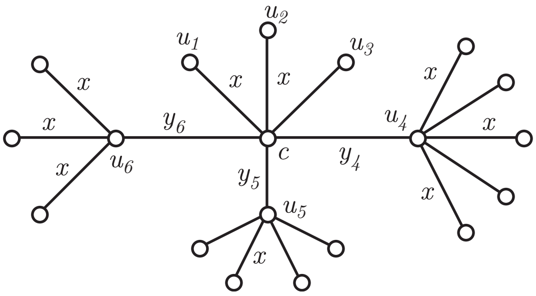

FIGURE 6. $TF(4;\ 1^3,6,5,4)$; An example of a tree of diameter 4.

Theorem 2.7. *Let* $n \geq 2$ *and consider the tree of diameter four* $G = TF(n; a_1, a_2, \cdots, a_n)$. *Also, let* $\sigma = 1 - 2n + \sum_{i=1}^{n} a_i$. *Then* G *is non-magic if and only if* $\sigma|(a_i - 2)$ *for some value of* $i = 1, 2, \cdots, n$.

Proof. As illustrated in Figure 6, in a magic labeling of G, we need to label all the leaves by $x \neq 0$ and cu_i by y_i. For each vertex u_i the condition $l^+(u_i) = x$, gives us $(a_i - 1)x + y_i = x$ or

$$(a_i - 2)x + y_i = 0. \tag{2.3}$$

Also, we require $l^+(c) = x$, or

$$\sum_{i=1}^{n} y_i = x. \tag{2.4}$$

If we add the equations in (2.3) and consider (2.4), we will get $(1 - 2n + \sum_{i=1}^{n} a_i)x = 0$, or

$$\sigma x = 0. \tag{2.5}$$

Now, if $\sigma|(a_i - 2)$ for some i, then (2.5) and (2.3) imply $y_i = 0$, which is not an acceptable magic label. Therefore, G is non-magic.

On the other hand, suppose σ does not divide $a_i - 2$ for all $i = 1, 2, \cdots, n$. Then $a_i \neq 2$ for all i and $\sigma \neq \pm 1$. We will consider the following two cases:

Case 1. If $\sigma \neq 0$, then we let $x = 1$. In this case the graph G will be σ-magic.

Case 2. If $\sigma = 0$, then Equation (2.5) is satisfied and if we choose $h = 1 + |\prod_{i=1}^{n}(a_i - 2)|$, then the graph G is h-magic.

Therefore, G is A-magic for the abelian group $A = \mathbb{Z}_h$. □

3. K-nonmagic Graphs

The complete bipartite graph $K(1, m)$ is called *star* and is denoted by $ST(m)$. The graph obtained by identifying one of the end vertices of star $ST(m)$ with one of the vertices of cycle C_3, as illustrated in the Figure 7, will be denote by $C_3(m)$.

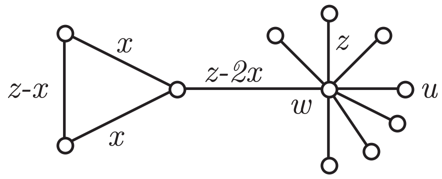

Figure 7. The graph $C_3(8)$ and its typical magic labeling.

The graph $C_3(m)$ has been studied in [13], and has been shown that $IM(\,C_3(1)\,) = IM(\,C_3(3)\,) = 2+2\mathbb{N}$, and $IM(\,C_3(2)\,) = \emptyset$. From now on we will assume that $m \geq 3$. Also, in any magic labeling of $C_3(m)$ one needs to have $l^+(w) = l^+(u)$ or $(m-1)z + z - 2x = z$, which implies

$$(m-1)z - 2x = 0. \tag{3.1}$$

Theorem 3.1. *If $m \geq 4$, then the graph $C_3(m)$ is $\mathbb{Z}$-magic.*

Proof. We observe that the choices of $x = m-1$ and $z = 2$ will work with with $l^+(w) = mz-2x = 2m - 2m + 2 = 2$. □

Theorem 3.2. *For any $m \geq 5$ the graph $C_3(m)$ is m-magic.*

Proof. The choices of $x = m-1$ and $z = 2$ will provide three distinct non-zero elements of $\mathbb{Z}_m$ with $l^+(w) = 2$. □

Theorem 3.3. *If $m = 2k+1$, then the graph $C_3(m)$ is 4-magic.*

Proof. We consider two cases:

Case 1. If $m = 4k+1$, then the choices of $x = 2$ and $z = 1$ will work with $l^+(w) = 4k+1 \equiv 1 \pmod 4$.

Case 2. If $m = 4k+3$, then the choices of $x = 3$, and $z = 1$ will work with $l^+(w) = (4k+2)+3 \equiv 1 \pmod 4$.

□

When m is even, the following theorem completely determines the integer-magic spectrum of $C_3(m)$.

Theorem 3.4. *If* $m = 2k \geq 4$, *then the integer-magic spectrum of* $C_3(m)$ *is*

$$\mathbb{N} - \{\ h\ :\ h|(2m-4),\ \ or\ \ h|(m-1),\ \ or\ \ h|(m-3)\ \}.$$

Proof. We will prove the theorem in four steps:

Step 1. If $h > 2m-4$, then $C_3(m)$ is h-magic. Because, the choices of $x = m-1$ and $z = 2$ will work with $l^{+}(w) = 2(m-1)+4-2m = 2$. Also, note that $z - 2x = 4-2m \not\equiv 0 \pmod{h}$.

Step 2. If $h > 1$ is any divisor of $2m-4$, then the graph $C_3(m)$ is not h-magic. It is enough to prove this statement for $h > 2$. Let $2m-4 = hq$, which implies that $m-1 = -(m-3)+hq$ or $m-1 \equiv -(m-3) \pmod{h}$. Now if we let $z = x+y$, the equation (3.1) becomes $(m-3)x + (m-1)y \equiv 0 \pmod{h}$, or $(m-3)(x-y) \equiv 0 \pmod{h}$. Since n is even, $\gcd(m-3, 2m-4) = 1$. But $\gcd(h, m-3) = 1$, as a result $x \equiv y \pmod{h}$, or $z = 2x$, which does not provide a valid labeling.

Step 3. If h is an odd divisor of either $m-1$ or $m-3$, then $C_3(m)$ is not h-magic. Here we observe that since $\gcd(m-3, m-1) = 1$, h cannot divide both $m-3$ and $m-1$. Without loss of generality, we may assume that h is a divisor of $m-3$ and $\gcd(h, m-1) = 1$. In this case, the equation (3.1) becomes $2(z-x) \equiv 0 \pmod{h}$, which does not provide a valid solution.

Step 4. Suppose $h \geq 3$ is not a divisor of any one of $m-1$, $m-3$, or $2m-4$, and let $\gcd(m-1, h) = d$, with $1 \leq d < h$. We will consider two cases:

(a) If $d = 1$, then choose $x = 1$. Then the equation (3.1) will become $(m-1)z \equiv 2 \pmod{h}$. Since $\gcd(m-1, h) = 1$, the equation has a non-zero solution $z \equiv 2(m-1)^{*} \pmod{h}$, where $(m-1)^{*}$ is the multiplicative inverse of $m-1$ in $\mathbb{Z}_h$. We observe that the conditions that h is not a divisor of $m-3$ and $2m-4$ will guarantee that $z \not\equiv 1, 2 \pmod{h}$.

(b) Suppose $\gcd(m-1, h) = d > 1$. Here d is odd and divides both $m-1$ and h. Therefore, from equation (3.1), $d|x$. Choose $x = d$ and let $m-1 = m^{'}d$, $h = h^{'}d$. Then equation (3.1) will become $m^{'}z \equiv 2 \pmod{h^{'}}$. Since $\gcd(m^{'}, h^{'}) = 1$, this equation has a non-zero solution $z \equiv 2\mu \pmod{h^{'}}$, where μ is the multiplicative inverse of $m^{'}$ in $\mathbb{Z}_{h^{'}}$. If $d|\mu$, we choose $z = 2\mu + h^{'}$; otherwise the choice of $z = 2\mu$ will provide a suitable magic labeling.

$\square$

Examples 3.5.

- **(a)** $IM(\,C_3(4)\,) = \mathbb{N} - \{1, 2, 3, 4\}$. Here, $m = 4$, and we need to exclude all the divisors $d \in \mathbb{N}$ of $2m - 4 = 4$, $m - 1 = 3$, and $m - 3 = 1$. The graph $C_3(4)$ is the only graph of the form $C_3(m)$ that is not h-magic for $h = 1, 2, 3, \cdots, m$; Because, by Theorem 3.2, for any $m \geq 5$, $C_3(m)$ is m-magic.
- **(b)** $IM(\,C_3(6)\,) = \mathbb{N} - \{1, 2, 3, 4, 5, 8\}$. Here, $m = 6$. We need to exclude all the divisors of $2m - 4 = 8$, $m - 1 = 5$, and $m - 3 = 3$.
- **(c)** $IM(\,C_3(8)\,) = \mathbb{N} - \{1, 2, 3, 4, 5, 6, 7, 12\}$. Here, $m = 8$. We need to exclude all the divisors of $2m - 4 = 12$, $m - 1 = 7$, and $m - 3 = 5$.
- **(d)** The integer-magic spectrum of $C_3(38)$ is
$$\mathbb{N} - \{h \in \mathbb{N}\ :\ 1 \leq h \leq 9\} \cup \{12, 18, 24, 35, 36, 37, 72\}.$$

Since $C_3(m)$ is $\mathbb{Z}$-magic, it is certainly h-magic for all sufficiently large values of $h \in \mathbb{N}$. It turns out that for certain choices of m, $C_3(m)$ is not h-magic for all positive integers in some initial interval $[1, K]$. Let us give a name for these particular graphs:

Definition 3.6. A $\mathbb{Z}$-magic graph G is said to be K-*nonmagic* if G is not h-magic for all values of $1 \leq h \leq K$.

Since, by Theorem 3.3, $C_3(2k+1)$ is is 4-magic, for the graph $C_3(m)$ to be K-nonmagic ($K \geq 4$) we require m to be even. Furthermore, in view of Theorem 3.2, m would have to be sufficiently large.

Theorem 3.7. *Given any $K \geq 2$, there exists a K-nonmagic graph G.*

Proof. As we observed in Example 3.5(a), the graph $C_3(4)$ is not h-magic for $h = 1, 2, 3, 4$. So we may assume that $K \geq 4$. Let $L = L_K = \operatorname{lcm}[2, 3, \cdots, K]$ and consider the graph $G = C_3(m)$, where $m = 2 + L/2$. Note that here L is divisible by 4, and so is $L + 4$, which implies that m is even. Also, any $h = 2, 3, \cdots, K$, is a divisor of $L = 2m - 4$. Therefore, the graph $C_3(m)$ is K-nonmagic. □

Note that the number m occurring in the proof of Theorem 3.7, may not be the smallest number with the desired property. To this end, let m_K denote the smallest integer greater than or equal to 4 such that $C_3(m_K)$ is not h-magic for $1 \leq h \leq K$. It follows from Example 3.5(a) that

$$m_2 = m_3 = m_4 = 4. \tag{3.2}$$

Furthermore, it follows from Theorem 3.2 and proof of Theorem 3.7 that, for $K \geq 5$, we have

$$K + 1 \leq m_K \leq 2 + L_K/2. \tag{3.3}$$

For large values of K the gap between the lower and the upper bound in (3.3) is enormous, and precise evaluation of m_K appears to be rather difficult. We can, however, improve the lower bound in (3.3), and we will show that the inequalities

$$2 + l_K \leq m_K \leq 2 + L_K/2 \tag{3.4}$$

certainly hold, where $l_K = \mathrm{lcm}\,[2, 3, \cdots, \lfloor K/2 \rfloor]$ (see Corollary 3.10). First however, we evaluate m_K for $5 \leq K \leq 10$.

Proposition 3.8. *$m_5 = 6,\ m_6 = m_7 = 8,\ m_8 = m_9 = 38,$ and $m_{10} = 542$.*

Note that, by (3.2) and (3.3), the equation $m_K = K$ has a unique solution $K = 4$. Furthermore, Proposition 3.8 yields two solutions to the equation $m_K = K + 1$, namely, $K = 5$ and 7. The problem of finding all solutions to this equation was raised in [13] Problem 3.11. We will show later that no further solutions to this equation exists.

Proof of Proposition 3.8. In Example 3.5(c) we showed that $IM(\,C_3(6)\,) = \mathbb{N} - \{1, 2, 3, 4, 5, 8\}$. Also, by 3.2, the graph $C_3(5)$ is 5-magic. Therefore, $m_5 = 6$. Since the technique of finding the m_K is analogous for $K = 6, 7, 8, 9, 10$, we will only demonstrate the case of $K = 10$.

Since for any $m \geq 5$ the graph $C_3(m)$ is m-magic, we need to find an even number $m \geq 12$ such that $4, 5, 6, \cdots, 10$ be divisors of one of $m-1,\ 2(m-2)$, or $m-3$. We observe that

1. 6 is an even number, can not be a divisor of $m-1$ or $m-3$. Therefore, 6 must be a divisor of $2(m-2)$, or $m \equiv 2 \pmod 3$.
2. Similarly, 10 is even and must be a divisor of $2(m-2)$, or $m \equiv 2 \pmod 5$, which at the same time will take care of number 5.
3. For number 8 we need $m \equiv 2 \pmod 4$.
4. For number 7, we need $m \equiv i \pmod 7$, with $i = 1, 2, 3$.
5. For number 9, we need to have $m \equiv j \pmod 9$, where $j = 1, 2, 3$. But because of (1), we must have $m \equiv 2 \pmod 9$.

To summarize the above discussion, we need to find an even number $m \geq 12$ such that

$$m \equiv 2 \pmod 4;$$

$$m \equiv 2 \pmod 5;$$

$$m \equiv i \pmod 7;$$

$$m \equiv 2 \pmod 9.$$

Using the Chinese Remainder Theorem, with equations $m \equiv a_k \pmod{h_k}$,

$$\begin{array}{rcccc} h_k: & 4 & 5 & 7 & 9 \\ a_k: & 2 & 2 & i & 2 \\ \mu_k: & 315 & 252 & 180 & 140 \\ b_k: & 3 & 3 & 3 & 2, \end{array} \tag{3.5}$$

where $\mu_j b_j \equiv 1 \pmod{m_j}$, we get $m = 1890 + 1512 + 540i + 560$, or $m \equiv 182 + 540i \pmod{1260}$, here $1260 = \prod m_k$. From this equation, all the possible answers are $m = 2, 542, 722 \pmod{1260}$. Therefore, $C_3(542)$ is a graph that is not h-magic for $h = 1, 2, 3, \cdots, 10$. Note that the least common multiple of the numbers $\{x : 2 \leq x \leq 10\}$ is $\mu = 2520$, and the equation $2m - 4 = 2520$ will provide the number presented in 3.7; namely, $m = 1262$. □

We now return to the lower bound estimate for m_K :

Theorem 3.9. *For $K \geq 4$ and any prime number $p \leq K/2$, let $\nu_p = \lfloor \log_p K \rfloor$. Then*

$$m_K \geq 2 + 2^{\nu_2 - 1} \prod_{3 \leq p \leq K/2} p^{\nu_p}. \tag{3.6}$$

Proof. By Theorem 3.4, m_K is the smallest even positive integer bigger than 3 such that every $h = 2, 3, \cdots, K$ divides of one of the numbers $m_K - 1$, $2(m_K - 2)$, or $m_K - 3$. Now if p is an odd prime with $p \leq K/2$, then we must have $2p | 2(m_K - 2)$, which gives $m_K \equiv 2 \pmod{p}$. It follows that

$$m_K \equiv 2 \pmod{p^{\nu_p}}. \tag{3.7}$$

Similarly, considering prime powers of 2 leads to $2^{\nu_2} | 2(m_K - 2)$, which gives

$$m_K \equiv 2 \pmod{2^{\nu_2 - 1}}. \tag{3.8}$$

The claim now follows from 3.7 and 3.8.

□

The lower bound of Theorem 3.9 does, in fact, agree with the values of m_K obtained in Proposition 3.8, for $K = 4, 6, 7, 8, 9$, as reader will readily verify. For larger K, however, we expect m_K to be of greater order of magnitude than the right side of (3.6).
Here are two corollaries of Theorem 3.9

Corollary 3.10. *If $l_K = \operatorname{lcm}[2, 3, \cdots, \lfloor K/2 \rfloor]$, then* $2 + l_K \leq m_K$.

Proof. This follows immediately from 3.6 since, by the definition of l_K, we have

$$l_K \Big| 2^{\nu_2 - 1} \prod_{3 \leq p \leq K/2} p^{\nu_p}. \tag{3.9}$$

□

Corollary 3.11. *For $K \geq 11$ we have $m_K > 2K$. In particular, the equation $m_K = K+1$ has two solutions $K = 5$ and 7.*

Proof. Since $p^{\nu_p} > K/p$, estimate 3.6 certainly yields, for $K \geq 11$,

$$m_K > \frac{K}{4}\frac{K}{3}\frac{K}{5} > 2K.$$

The second claim of corollary now follows from Proposition 3.8. □

As observed in the case of $K = 10$ (Proposition 3.8), $m_{10} = 542$ is the smallest number for which $C_3(542)$ is 10-nonmagic. But there are other solutions as well. In fact, we noticed that $m_{10} = 2,\ 542,\ 722 \pmod{1260}$. Here, there are three equivalence classes modulus $1260 = \frac{1}{2}\mathrm{lcm}\,[2,3,\cdots,10] = \frac{1}{2}L_{10}$. The number of equivalence classes depends on the frequency of prime numbers in the interval $(K/2,\ K]$. In general, if there are r prime numbers in the interval $(K/2,\ K]$, then there would be 3^r equivalence classes modulus $\frac{1}{2}\mathrm{lcm}\,[2,3,\cdots,K]$. In the following table, the smallest number m_K and some other related facts are summarized.

A few non-magic numbers

K	m_K	Modulus	Number of equivalence Classes
5	6	30	3^2
6	8	30	3
7	8	210	3^2
8	38	420	3^2
9	38	1260	3^2
10	542	1260	3
11	542	13860	3^2
12	542	13860	3^2
13	9362	180180	3^3
14	16382	180180	3^2
15	16382	180180	3^2
16	80642	360360	3^2
17	246962	6126120	3^3
18	246962	6126120	3^3
19	1081082	116396280	3^4
20	1081082	116396280	3^4

We conclude this paper by the following two problems:

Problem 3.12. *In Theorems 2.4 and 2.7, the non-magic trees of diameter at most four were characterized. Find a characterization of non-magic trees of diameter at least five.*

Problem 3.13. *In this paper, the graphs $C_3(m)$ were utilized to show the existence of K-nonmagic graphs. Find other classes of K-nonmagic graphs.*

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