# Integer-Magic Spectra of Trees with Diameters at most Four

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**Abstract**

For any $k \in \mathbb{N}$, a graph $G = (V, E)$ is said to be $\mathbb{Z}_k$-magic if there exists a labeling $l : E(G) \longrightarrow \mathbb{Z}_k - \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \longrightarrow \mathbb{Z}_k$ defined by

$$l^+(v) = \sum_{u \in N(v)} l(uv)$$

is a constant map. For a given graph $G$, the set of all $k \in \mathbb{Z}_+$ for which $G$ is $\mathbb{Z}_k$-magic is called the integer-magic spectrum of $G$ and is denoted by $IM(G)$. In this paper we will consider trees whose diameters are at most 4 and will determine their integer-magic spectra.

**Keywords:** Integer-magic Spectrum, Magic, and Non-magic graphs.

**AMS Subject Classification:** 05C15.

## 1 Introduction

For any abelian group $A$, written additively, any mapping $l : E(G) \longrightarrow A - \{0\}$ is called a *labeling*, or edge-labeling. Given an edge-labeling of G one can introduce a vertex labeling $l^+ : V(G) \longrightarrow A$ as follows:

$$l^+(v) = \sum_{u \in N(v)} l(uv),$$

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where $N(v)$ denotes the set of all vertices of $G$ that are adjacent with $v$. A graph $G$ is said to be *$A$-magic* if there is a labeling $l : E(G) \longrightarrow A - \{0\}$ such that for each vertex $v$, the sum of the labels of the edges incident with $v$ are all equal to the same constant; that is, $l^+(v) = c$ for some fixed $c \in A$. We will call $< G, l >$ an $A$-magic graph with sum $c$. In general, a graph $G$ may admit more than one labeling to become an $A$-magic graph; for example, if $|A| > 2$ and $l : E(G) \longrightarrow A - \{0\}$ is a magic labeling of $G$ with sum $c$, then $\lambda : E(G) \longrightarrow A - \{0\}$, the inverse labeling of $l$, defined by $\lambda(uv) = -l(uv)$ will provide another magic labeling of $G$ with sum $-c$.

The original concept of an $A$-magic graph is due to J. Sedlacek [21, 22], who defined it to be a graph with a real-valued edge labeling such that

1. distinct edges have distinct nonnegative labels; and
2. the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Given a graph $G$, the problem of deciding whether $G$ admits a magic labeling is equivalent to the problem of deciding whether a set of linear homogeneous Diophantine equations has a solution [23]. At present, given an abelian group, no general efficient algorithm is known for finding magic labelings for general graphs.

When $A = \mathbb{Z}$, the $\mathbb{Z}$-magic graphs were considered in Stanley [23, 24], he pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. When the group is $\mathbb{Z}_k$, we shall refer to the $\mathbb{Z}_k$-magic graph as *$k$-magic*. Graphs which are $k$-magic had been studied in [6, 9, 10, 12, 14, 16, 18]. For convenience, we will use the notation 1-magic instead of $\mathbb{Z}$-magic. Doob [2, 3, 4], also considered $A$-magic graphs where $A$ is an abelian group. He determined which wheels are $\mathbb{Z}$-magic.

A graph $G = (V, E)$ is called *fully magic* [14, 16] if it is $A$-magic for every abelian group $A$, and it is called *non-magic* if for every abelian group $A$ it is not $A$-magic. Also, a graph $G$ is said to be $\mathbb{N}$-magic if there exists a labeling $l : E(G) \rightarrow \mathbb{N}$ such that $l^+(v)$ is a constant, for every $v \in V(G)$. It is well-known that a graph $G$ is $\mathbb{N}$-magic if and only if each edge of $G$ is contained in a 1-factor (a perfect matching) or a $\{1, 2\}$-factor [11, 20, 29]. Berge [1] called a graph *regularisable* if a regular multigraph could be obtained from $G$ by adding edges parallel to the edges of $G$. In fact, a graph is regularisable if and only if it is $\mathbb{N}$-magic. For $\mathbb{N}$-magic graphs, readers are referred to [7, 8, 9, 10, 12, 27, 24, 25, 26]. The notion of $\mathbb{Z}$-magic is weaker than $\mathbb{N}$-magic. Figure 1 shows a graph which is $\mathbb{Z}$-magic but not $\mathbb{N}$-magic.

**Observation 1.1.** *For any $n \geq 3$, the path of order $n$ is non-magic.*

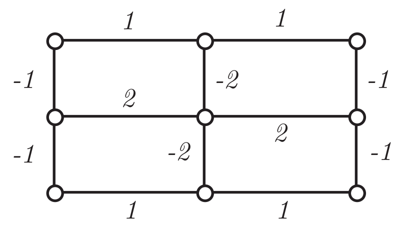


Figure 1: The graph $P_3 \times P_3$ is $\mathbb{Z}$-magic but is not $\mathbb{N}$-magic.

**Observation 1.2.** *Any graph with a pendant path of length two is non-magic.*

In this paper, we will denote the set of positive integers by $\mathbb{N}$, and for any $k > 0$,

$$k\mathbb{N} = \{\ kn\ :\ n \in \mathbb{N}\ \}, \quad \text{also} \quad k + \mathbb{N} = \{\ k + n\ :\ n \in \mathbb{N}\ \}.$$

For a given graph $G$ the set of all positive integers $h$ for which $G$ is $\mathbb{Z}_h$-magic (or simply $h$-magic) is called the *integer-magic spectrum* of $G$ and is denoted by $IM(G)$. Since any regular graph is fully magic, then it is $h$-magic for all positive integers $h \geq 2$; therefore, $IM(G) = \mathbb{N}$. In what follows we will consider trees whose diameters are at most 4 and will determine their integer-magic spectra.

## 2 Trees with diameters two; Stars

For any $n \geq 1$, the complete bipartite graph $K(1, n)$ is called a *star* and is denoted by $ST(n)$. Note that $K(1, 1)$ is the same as $P_2$, the path of order two, and it is fully magic. Also, $K(1, 2)$ is the same as $P_3$, the path of order three, which is non-magic. To study the integer-magic spectrum of $ST(n)$ we will assume that $n \geq 3$.

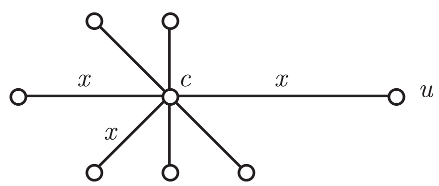


Figure 2: A typical labeling of $ST(n)$.

**Theorem 2.1.** *Let $n \geq 3$, and $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $n-1$. Then* $IM(\,ST(n)\,) = \bigcup_{i=1}^{k} p_i\mathbb{N}$.

*Proof.* In a typical magic labeling of $ST(n)$, as illustrated in the Figure 2, all the edges are labeled with the same element $x$ of the group $\mathbb{Z}_h$. The required condition is $l^+(c) = l^+(u)$ or

$$(n-1)x \equiv 0 \pmod{h}. \tag{1}$$

If $\gcd(n-1, h) = 1$, then Equation (1) will become $x \equiv 0 \pmod{h}$, which is not an acceptable solution. Therefore, we need $\gcd(n-1, h) = \delta > 1$, which implies $h \in \bigcup_{i=1}^{k} p_i\mathbb{N}$, and Equation (1) will have a non-zero solution for $x$. On the other hand, let $p$ be a prime factor of $n-1$ and let $h \in p\mathbb{N}$. Then the choice of $x = \frac{h}{p}$ will work; Because, $l^+(c) = nx = (n-1)\frac{h}{p} + \frac{h}{p} \equiv \frac{h}{p} \pmod{h}$. $\square$

**Examples 2.2.**

*(a)* $IM(\,ST(33)\,) = 2\mathbb{N}$; *here* $n-1 = 32 = 2^5$.

*(b)* $IM(\,ST(25)\,) = 2\mathbb{N} \cup 3\mathbb{N}$; *here* $n-1 = 24 = 2^3 \times 3$.

*(c)* $IM(\,ST(361)\,) = 2\mathbb{N} \cup 3\mathbb{N} \cup 5\mathbb{N}$; *here* $n-1 = 360 = 2^3 \times 3^2 \times 5$.

## 3 Trees with diameters three; Double-Stars

Trees with diameter 3 are called double-stars. These graphs have two central vertices $u$ and $v$ plus leaves. We will use $DS(m,n)$ to denote the double-star whose two central vertices have degrees $m$ and $n$, respectively. By the Observation 1.1, if $m = 2$ or $n = 2$, then $DS(m,n)$ is non-magic, therefore the integer-magic spectrum will be $\emptyset$. As a result, in what follows, we will assume that $m \geq n \geq 3$. Moreover, being a tree, $DS(m,n)$ is 2-magic if and only if $m$ and $n$ are odd numbers.

Note that in any magic labeling of a graph the end-edges (edges incident with the end-vertices) are labeled with the same element of the group $A$. Therefore, for any magic labeling of a double star, as illustrated in Figure 3, we use at most two non-zero group elements $x$ and $y$.

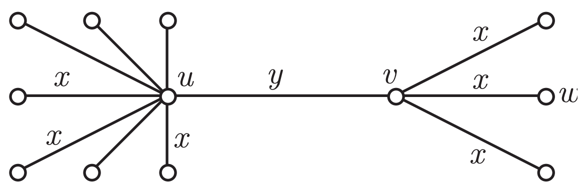


Figure 3: A typical magic labeling of $DS(m,n)$.

In particular, we need to have $l^+(u) = l^+(v)$ and $l^+(v) = l^+(w)$. These equations, when we are using $\mathbb{Z}_h$ will translate to:

$$(m-n)x \equiv 0 \pmod{h}; \tag{2}$$

$$(n-2)x + y \equiv 0 \pmod{h}. \tag{3}$$

**Theorem 3.1.** *The graph $DS(m,n)$ is $\mathbb{Z}$-magic (or 1-magic) if and only if $m = n$.*

*Proof.* If $DS(m,n)$ is $\mathbb{Z}$-magic, then Equation (2) would imply that $(m-n)x = 0$. Since $x$ is non-zero, then we will have $m = n$. Conversely, if $m = n$, then the choices of $x = 1$ and $y = 2 - m$ provide a magic labeling with $l^+ \equiv 1$. □

**Theorem 3.2.** $IM(\,DS(m,m)\,) = \mathbb{N} - \{\ h \in \mathbb{N}\ :\ h > 1\ \&\ h|(m-2)\ \}$.

*Proof.* By the Theorem 3.1, $DS(m,m)$ is 1-magic and if $h > m-2$, then the choices of $x = 1,\ y = h - m + 2$ will work with $l^+ \equiv 1$. Now assume that $1 < h \leq m-2$. Since $m = n$, Equation (2) holds. It is enough to show that (3) is true. Note that $DS(m,m)$ is $h$-magic if and only if $h$ is not a divisor of $(m-2)$. Because, if $h|(m-2)$, then (3) becomes $y \equiv 0 \pmod{h}$, which is not an acceptable answer. On the other hand, if $h$ does not divide $(m-2)$, then the choices of $x = 1,\ y = 2 - m \pmod{h}$ will work with $l^+ \equiv 1$. □

**Examples 3.3.**

(a) $IM(\,DS(3,3)\,) = \mathbb{N}$; here $m-2 = 1$ does not have any divisor bigger than 1. In fact, $DS(3,3)$ is the only double-star whose integer-magic spectrum is $\mathbb{N}$.

(b) $IM(\,DS(11,11)\,) = \mathbb{N} - \{3,9\}$; here $m-2 = 9$, and we need to exclude its divisors that are bigger than one: namely, $3, 9$.

(c) $IM(\,DS(26,26)\,) = \mathbb{N} - \{2,3,4,6,8,12,24\}$; here $m-2 = 24$, and we need to exclude its divisors that are bigger than 1.

**Theorem 3.4.** *Let $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ be the prime factorizations of $m-n$ and $n-2$, respectively. Then $IM(\,DS(m,n)\,) = \cup_{i=1}^{k} A_i$, where*

$$A_i = \begin{cases} p_i^{1+\beta_i}\mathbb{N} & \text{if } \alpha_i > \beta_i \geq 0; \\ \emptyset & \text{if } \beta_i \geq \alpha_i \geq 0. \end{cases}$$

*Proof.* We will prove the statement through four steps.

Step 1. Note that if $m-n=1$ or $m-n$ be a divisor of $n-2$ or $m-2$, then $IM(\,DS(m,n)\,)=\emptyset$. Because, $m-n=1$ will convert Equation (2) to $x=0$, which is not an acceptable solution. Also, if $m-2=q(m-n)$, then Equation (3) becomes $y=-(m-2)x=-q(m-n)x=0$, again an unacceptable solution. This is consistent with the statement of the theorem, for in either cases $\beta_i \geq \alpha_i$.

Therefore, we may assume that $m-n>1$ and that $m-n$ is not a divisor of $n-2$.

Step 2. Let $p$ be a prime number, $\alpha > \beta$, $p^\alpha|(m-n)$, and $p^\beta|(n-2)$, but $p^{\beta+1}$ does not divide $(n-2)$. Then the graph $DS(m,n)$ is $p^{\beta+1}$-magic. Here, since $m-n \equiv 0 \pmod{p^\alpha}$, Equation (2) holds. Choose $x=1$. Note that $y \equiv 2-n \equiv\ 2-m \pmod{p^{\beta+1}}$ is non-zero and these labels work with $l^+ \equiv 1;\ l^+(v)=m-1+2-n \equiv m-n+1 \equiv 1 (\mathrm{mod}\ p^{\beta+1})$.

Step 3. If the graph $DS(m,n)$ is $h$-magic, then for every $r \in \mathbb{N}$ the graph is $hr$-magic. To see this, we simply observe that if $x,y$ are the non-zero labels modulo $h$, then $rx, ry$ will be non-zero modulo $hr$ and will be valid magic labelings. Combination of these steps shows that the integer-magic spectrum of $DS(m,n)$ contains $p^{\beta+1}\mathbb{N}$.

Step 4. The previous steps show that $\cup_{i=1}^k A_i \subset IM(\,DS(m,n)\,)$. To show that

$$IM(\,DS(m,n)\,)\ \subset \cup_{i=1}^k A_i,$$

suppose $DS(m,n)$ is $h$-magic. Then from Equation (3), $h$ does not divide $(n-2)x$; otherwise, $y$ will be zero. But from Equation (2), $h|(m-n)x$. Therefore, in the prime factorization of $h$ there exists a prime factor $p^\gamma$ with the property that $p^\gamma|(m-n)x$, while $p^\gamma$ does not divide $(n-2)x$. Choose $\beta \geq 0$ such that $p^{\beta+1}|(m-n)$, but $p^{\beta+1}$ does not divide $n-2$. By step 2, the graph $DS(m,n)$ is $p^{\beta+1}$-magic, and by step 3, $h \in p^{\beta+1}\mathbb{N}$. This completes the proof of the theorem.

□

**Corollary 3.5.** *$IM(\,DS(m,n)\,)=\emptyset$ if and only if $(m-n)|(n-2)$.*

*Proof.* If $IM(\,DS(m,n)\,)=\emptyset$, then by 3.4, $\beta_i \geq \alpha_i$ $(1 \leq i \leq k)$ and hence $(m-n)|(n-2)$. Conversely, if $(m-n)$ divides $n-2$, then $y \equiv -(n-2)x \equiv 0$, not an acceptable solution. □

**Corollary 3.6.** *If* $|m-n|=1$, *then* $DS(m,n)$ *is non-magic.*

**Examples 3.7.**

(a) $IM(\,DS(9,3)\,) = 2\mathbb{N} \cup 3\mathbb{N}$; here $m-n=6=2\times 3$, while $n-2=1$.

(b) $IM(\,DS(6,4)\,) = \emptyset$; here $m-n=2$ is a divisor of $n-2=2$.

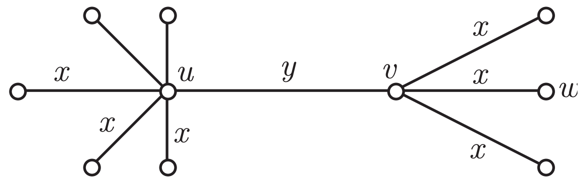


Figure 4: $IM(\,DS(6,4)\,) = \emptyset$.

(c) $IM(\,DS(28,4)\,) = 4\mathbb{N} \cup 3\mathbb{N}$; here $m-n=24=2^3\times 3$, while $n-2=2$.

(d) $IM(\,DS(16,10)\,) = 3\mathbb{N}$; here $m-n=6=2\times 3$, while $n-2=2^3$.

(e) $IM(\,DS(20,14)\,) = \emptyset$; here $m-n=6$ is a divisor of $n-2=12$.

# 4 Trees of diameter four

**Definition 4.1.** A tree of diameter four, denoted by $TF(n;a_1,a_2,\cdots,a_n)$, consists of $n$ stars $ST(a_1),ST(a_2),\cdots,ST(a_n)$ one of their edges is incident with a common vertex. The common vertex will be called the center of the tree and will be denoted by $c$.
In other words, $TF(n;a_1,a_2,\cdots,a_n)$ is a tree with center-vertex $c$, in which $n$ edges $\{cu_1,cu_2,\cdots,cu_n\}$ are emanated from $c$, and $\deg(u_i)=a_i$ for each $i=1,2,\cdots,n$, as illustrated in the Figure 5. In order to have a tree of diameter four, one needs $n\geq 2$ and $a_i\geq 2$ for at least two values of $i$.

**Observation 4.2.** *If one of* $a_1,a_2,\cdots,a_n$ *is* 2, *then* $IM(\,TF(n;\ a_1,a_2,\cdots,a_n)) = \emptyset$.

**Observation 4.3.** *Let* $b_1,b_2,\cdots,b_n$ *be any permutation of* $a_1,a_2,\cdots,a_n$. *Then* $TF(n;\ a_1,a_2,\cdots,a_n)$ *is isomorphic with* $TF(n;\ b_1,b_2,\cdots,b_n)$.

As a result of these two observations, in any tree $TF(n;\ a_1,a_2,\cdots,a_n)$ with diameter four, we may assume that $1<n,\ a_1\leq a_2\leq\cdots\leq a_n$, and $a_i\neq 2$.

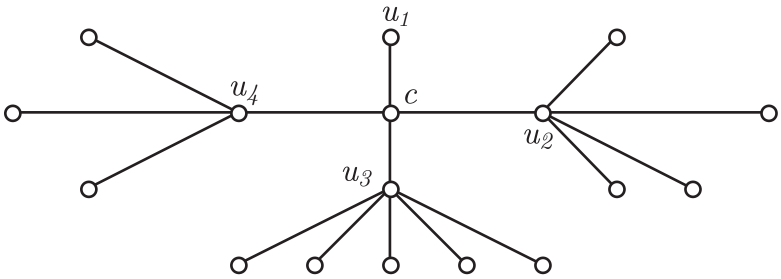


Figure 5: $TF(4;\ 1,5,6,4)$; An example of a tree of diameter 4.

**Theorem 4.4.** *Suppose* $3 \leq m < n$ *and let* $m+n-3 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ *be the prime factorization of* $m+n-3$. *Then*

$$IM(\,TF(2;\, m,n)\,) = \bigcup_{i=1}^{k} p_i\mathbb{N} - \{\ d \mid d \ \textit{ is a divisor of } \ m-1,\, m-2,\, n-1,\ \textit{ or } ,n-2\ \}.$$

*Proof.* For any magic labeling of this graph, as illustrated in the Figure 6, one needs two distinct non-zero elements $x, y \in \mathbb{Z}_h$.

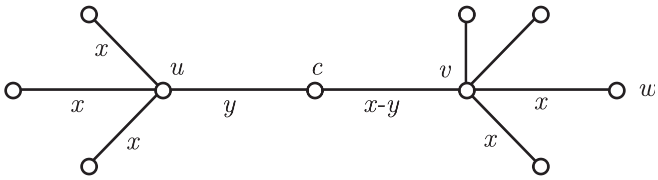


Figure 6: A typical magic labeling of $SF(2;\, 4,5)$. Here $x \neq y$.

The condition $l^{+}(u) = x$ implies

$$(m-1)x + y \equiv x \pmod{h}. \tag{4}$$

Since $x \neq y$, $h$ cannot be a divisor of $m-1$. Also Equation (4) can be written as $y \equiv -(m-2)x \pmod{h}$, which implies that $h$ cannot be a divisor of $m-2$ either. Similarly, from $l^{+}(v) = x$ we will have

$$(n-1)x + x - y \equiv x \pmod{h}. \tag{5}$$

If we add the two Equations (4) and (5) we get

$$(m+n-3)x \equiv 0 \pmod{h}. \tag{6}$$

This last equation has a non-zero solution for $x$ if and only if $\gcd(m+n-3,\, h) > 1$; that is, $h \in \bigcup_{i=1}^{k} p_i\mathbb{N}$. □

**Corollary 4.5.** *Let $m \geq 3$, and $2m-3 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $2m-3$. Then $IM(\,TF(2;\, m, m)\,) = \bigcup_{i=1}^{k} p_i \mathbb{N}$.*

*Proof.* Here $m = n$ and

$$\bigcup_{i=1}^{k} p_i \mathbb{N} \bigcap \{\ d \mid d \text{ is a divisor of } m-1 \text{ or } m-2\ \} = \emptyset.$$

□

**Theorem 4.6.** *Consider the tree of diameter four $G = TF(n;\, a_1, a_2, \cdots, a_n)$ $(n \geq 3)$, and let $\pm p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of $\sigma = 1 - 2n + \sum_{i=1}^{n} a_i$. Then the integer-magic spectrum of $G = TF(n;\, a_1, a_2, \cdots, a_n)$ is*

$$IM(G) = \begin{cases} \emptyset & \text{if } \sigma | (a_i - 2) \text{ for some } i > k \\ \mathbb{N} - D & \text{if } \sigma = 0 \\ \bigcup_{i=1}^{k} p_i \mathbb{N} - D & \text{otherwise,} \end{cases}$$

*where $D = \{\ d \mid d$ is a divisor of $a_i - 2$ for some $i$ $(k+1 \leq i \leq n)\ \}$.*

*Proof.* First, note that if $a_i = 2$ for some $i$, then $G$ is non-magic, and in this case $\sigma | (a_i - 2)$, which is consistent with the statement of the Theorem. Therefore, we may assume that $a_i \neq 2$. Let $a_1 = a_2 = \cdots = a_k = 1$ and $a_i \geq 3$ $(k+1 \leq i \leq n)$.

In labeling of $G = TF(n;\, a_1, a_2, \cdots, a_n)$, we will use $x \in \mathbb{Z}_h$ for all the leaves and if $a_i \geq 3$, we will use $y_i \in \mathbb{Z}_h$ to label the edge $cu_i$ (See the Figure 5). The condition $l^+(u_i) = x$ implies $(a_i - 1)x + y_i \equiv x \pmod{h}$ or

$$y_i \equiv -(a_i - 2)x \pmod{h} \quad (k+1 \leq i \leq n). \tag{7}$$

Since $y_i$ is a non-zero element of the group $\mathbb{Z}_h$, $h$ cannot be a divisor of $a_i - 2$. Also, the condition $l^+(c) = x$ implies that $kx + y_{k+1} + y_{k+2} + \cdots + y_n \equiv x \pmod{h}$, or

$$\sum_{i=k+1}^{n} y_i \equiv (1-k)x \pmod{h}. \tag{8}$$

Now if we add the Equations (7) for $i = k+1, k+2, \cdots, n$, we will get $\sum_{i=k+1}^{n} y_i \equiv -\sum_{i=k+1}^{n} (a_i - 2)x \pmod{h}$, which together with (8) will give us $(1 - 2n + \sum_{i=1}^{n} a_i)x \equiv 0 \pmod{h}$, or

$$\sigma x \equiv 0 \pmod{h}. \tag{9}$$

Now we will consider the following cases:

Case 1. If for some value of $i$, $\sigma|(a_i-2)$, then $a_i-2=q\sigma$ and Equation (7) gives us $y_i \equiv -q\sigma x \equiv 0 \pmod{h}$, which is not an acceptable answer. Therefore, $IM(G)=\emptyset$.

Case 2. If $\sigma=0$, Equation (9) is satisfied for all values of $h$, and one only needs to exclude divisors of $a_i-2$ to assure that Equations (7) will provide non-zero solutions for $y_i$. Therefore, $IM(G)=\mathbb{N}-D$.

Case 3. Suppose $\sigma \neq 0$ and $\sigma$ is not divisor of any of $a_i-2$. Then Equation (9) has non-zero solution for $x$ if and only if $\gcd(\sigma,\ h)>1$; that is, $h \in \bigcup_{i=1}^{k} p_i\mathbb{N}$. However, we need to exclude divisors of $a_i-2$ to make sure that Equations (7) will provide non-zero solutions for $y_i$. Therefore, $IM(G)=\bigcup_{i=1}^{k} p_i\mathbb{N}-D$.

This complete the proof of the Theorem. □

**Corollary 4.7.** *With the notation of Theorem 4.6, if* $a_1 \geq 3$, *then*

$$IM(G)=\bigcup_{i=1}^{k} p_i\mathbb{N}-\{\ d \mid d \ \textit{is a divisor of} \ a_i-2 \ (1\leq i\leq n)\ \}.$$

*Proof.* We observe that if $a_1 \geq 3$, then $\sigma = 1-2n+\sum_{i=1}^{n} a_i \geq 1-2n+3(n-1)+a_i > a_i-2$. Therefore, $\sigma \neq 0$ and it can not be a divisor of $a_i-2$. □

**Corollary 4.8.** *In* $TF(n;\ a_1,a_2,\cdots,a_n)$ *let* $a_1=a_2=\cdots=a_n=m\geq 3$; *that is choose* $n$ *copies of* $ST(m)$ *and identify one of their end vertices. Also let* $mn-2n+1=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ *be the prime factorization of* $mn-2n+1$. *Then*

$$IM(\,TF(n;\,m,m,\cdots,m)\,)=\{\,h \mid \ \gcd(mn-2n+1,h)>1\,\}=\bigcup_{i=1}^{k} p_i\mathbb{N}.$$

**Examples 4.9.**

(a) $IM(\,TF(2;\,4,4)\,)=5\mathbb{N}$; here $2m-3=5$.

(b) $IM(\,TF(2;\,6,17)\,)=2\mathbb{N}\cup 3\mathbb{N}-\{2,4,5,8,15,16\}$; here $a_1+a_2-3=20$, and we need to exclude the divisors of $a_i-1$, $a_i-2$.

(c) $IM(\,TF(2;\,9,24)\,)=2\mathbb{N}\cup 3\mathbb{N}\cup 5\mathbb{N}-\{2,4,8,22\}$; here $a_1+a_2-3=30$.

(d) $IM(\,TF(3;\,5,5,5)\,)=2\mathbb{N}\cup 5\mathbb{N}$; here $3m-5=10=2\times 5$.

(e) $IM(\,TF(3;\,3,5,9)\,)=2\mathbb{N}\cup 3\mathbb{N}-\{3\}$; here $a_1+a_2+a_3-5=12$ and 3 is a divisor of $a_2-2$.

(f) $IM(\,TF(4;\,1,1,3,5)\,) = \emptyset$; here $\sigma = a_1 + a_2 + a_3 + a_4 - 7 = 3$ and $\sigma = 3$ is a divisor of $a_4 - 2$.

(g) $IM(\,TF(6;\,1,1,1,3,3,4)\,) = \emptyset$; here $\sigma = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 - 11 = 2$, which is a divisor of $a_6 - 2$.

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