INTEGER-MAGIC SPECTRA OF CYCLE RELATED GRAPHS

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Abstract. For any $h \in \mathbb{N}$, a graph $G = (V, E)$ is said to be h-magic if there exists a labeling $l : E(G) \longrightarrow \mathbb{Z}_h - \{0\}$ such that the induced vertex set labeling $l^+ : V(G) \longrightarrow \mathbb{Z}_h$ defined by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. For a given graph G, the set of all $h \in \mathbb{Z}_+$ for which G is h-magic is called the integer-magic spectrum of G and is denoted by $IM(G)$. In this paper, the integer-magic spectra of certain classes of cycle related graphs will be determined.

Key Words: magic, non-magic, integer-magic spectrum.

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1. Introduction

In this paper all graphs are connected, finite, simple, and undirected. For an abelian group A, written additively, any mapping $l : E(G) \longrightarrow A - \{0\}$ is called a *labeling*. Given a labeling on the edge set of G one can introduce a vertex set labeling $l^+ : V(G) \longrightarrow A$ by

$$l^+(v) = \sum_{uv \in E(G)} l(uv).$$

A graph G is said to be *A-magic* if there is a labeling $l : E(G) \longrightarrow A - \{0\}$ such that for each vertex v, the sum of the labels of the edges incident with v are all equal to the same constant; that is, $l^+(v) = c$ for some fixed $c \in A$. In general, a graph G may admit more than one labeling to become A-magic; for example, if $|A| > 2$ and $l : E(G) \longrightarrow A - \{0\}$ is a magic labeling of G with sum c, then $l : E(G) \longrightarrow A - \{0\}$, the *inverse labeling* of l, defined by $l(uv) = -l(uv)$ will provide another magic labeling of G with sum $-c$. A graph $G = (V, E)$ is called *fully magic* if it is A-magic for every abelian group A. For example, every regular graph is fully magic. A graph $G=(V,E)$ is called *non-magic* if for every abelian group A, the graph is not A-magic. The most obvious class of non-magic graphs is P_n $(n \geq 3)$, the path of order n. As a result, any graph with a pendant path of length $n \geq 3$ would be non-magic. Here is another example of a non-magic graph: Consider the graph H Figure 1. Given any abelian group A, a typical magic labeling of H is illustrated in that figure. Since $l^+(u) = x \neq 0$ and $l^+(v) = 0$, then H is not A-magic. This fact can be generalized as follows:

Observation 1.1. *Every even cycle C_n with $2k+1\,(< n)$ consecutive pendants is non-magic.*

Observation 1.2. *Every odd cycle C_n with $2k\,(< n)$ consecutive pendants is non-magic.*

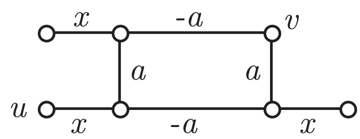

FIGURE 1. An example of a non-magic graph.

Certain classes of non-magic graphs are presented in [1].

The original concept of A-magic graph is due to J. Sedlacek [9, 10, 11], who defined it to be a graph with a real-valued edge labeling such that

(1) distinct edges have distinct nonnegative labels; and
(2) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Jenzy and Trenkler [3] proved that a graph G is magic if and only if every edge of G is contained in a $(1-2)$-factor. $\mathbb{Z}$-magic graphs were considered by Stanley [12, 13], who pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. Recently, there has been considerable research articles in graph labeling, interested readers are directed to [2, 14]. For convenience, the notation 1-magic will be used to indicate $\mathbb{Z}$-magic and $\mathbb{Z}_h$-magic graphs will be referred to as h-magic graphs. Clearly, if a graph is h-magic, it is not necessarily k-magic $(h \neq k)$.

Definition 1.3. *For a given graph G the set of all positive integers h for which G is h-magic is called the* integer-magic spectrum *of G and is denoted by $IM(G)$.*

Since any regular graph is fully magic, then it is h-magic for all positive integers $h \geq 2$; therefore, $IM(G) = \mathbb{N}$. On the other hand, the graph H, Figure 1, is non-magic, hence $IM(H) = \emptyset$. The integer-magic spectra of certain classes of graphs resulted by the amalgamation of cycles and stars have already been identified [4], and in [5] the integer-magic spectra of the trees of diameter at most four have been completely characterized. Also, the integer-magic spectra of some other graphs have been studied in [6, 7, 8].

In the following sections the integer-magic spectra of certain classes of cycle related graphs will be determined. There are different classes of cycle related graphs that have been studied for variety of labeling purposes. J. Gallian [2] has a nice collection of such graphs. First, two useful observations:

Observation 1.4. *Suppose G has a magic labeling in $\mathbb{Z}_h$ such that the sum of labels adjacent with all vertices is $c \leq h$. Then G is k-magic for all $k \geq h$.*

Observation 1.5. *If a graph G has a $\mathbb{Z}$-magic labeling $\ell : E(G) \to \mathbb{Z}$ and $k \in \mathbb{N}$ does not divide $\ell(e)$ for every $e \in E(G)$, then G is k-magic.*

Observation 1.6. *In any magic labeling of of a cycle the edges should alternatively be labeled the same elements of the group.*

Proof. Let $u_1,\ u_2,\ u_3$, and u_4 be the four consecutive vertices of a cycle. The requirement of $l(u_1u_2) + l(u_2u_3) = l(u_2u_3) + l(u_3u_4)$ implies that $l(u_1u_2) = l(u_3u_4)$. □

2. Wheels

Wheels are defined to be $W_n = C_n + K_1$, where C_n is the cycle of order $n \geq 3$.

Theorem 2.1. *If $n \geq 3$, then $IM(W_n) = \mathbb{N} - \{1 + (-1)^n\}$.*

Proof. We will consider two cases:

Case I. $n = 2k + 1$ is odd. We observe that the degree set of W_{2k+1} is $\{3, 2k + 1\}$, hence it is h-magic for all even numbers h; we simply label all the edges by $h/2$. Also, if $h > k$, then we label all the cycle edges by k and spokes by 1. This is a magic labeling of W_{2k+1} with sum $n = 2k + 1$.

Now, we may assume that h is odd and is at most k. If $\gcd(k,\, h) = \delta,\ 1 \leq \delta < h$, then we label the cycle edges by δ and spokes by x, where x is the nonzero solution of the equation $kx \equiv \delta \pmod{h}$. This provides a magic labeling of W_{2k+1} with sum $x + 2\delta$.

Finally, if $h|k$, we label $h + 1$ consecutive spokes by 1 and the rest of them by $h - 1$. For cycle edges, we label those that are adjacent to the spokes labeled $h - 1$ by 1 the remaining by $h - 1$

and 1, alternatively. This would be a magic labeling of W_{2k+1} with sum 1. Therefore, W_{2k+1} is h-magic for all $h \geq 1$; that is $IM(W_{2k+1}) = \mathbb{N}$.

Case II. $n = 2k$ is even. We observe that the degree set of W_{2k} is $\{3, 2k\}$, hence it cannot be 2-magic. Next we label all the spokes by x and the cycle edges by a, b, alternatively. The requirement of having the same number for the sum of the edges incident with vertices will provide the equation

$$(2k-1)x \equiv a + b \pmod{h}. \tag{2.1}$$

If $\gcd(2k-1,\, h) = \delta \geq 3$, then we choose $a = 1,\ b = -1$, and $x = h/\delta$. this would be a magic labeling of W_{2k} with sum x.

If $\gcd(2k-1,\, h) = 1$, then we choose $a = b = 1$ and notice that the equation $(2k-1)x \equiv 2 \pmod{h}$ has a nonzero solution for x. We label all the spokes with this x, the result is a magic labeling of W_{2k} in $\mathbb{Z}_h$ with sum $x + 2$. Therefore, $IM(W_{2k}) = \mathbb{N} - \{2\}$. □

3. Fans

If we join a vertex of C_n to all other vertices, the resulting graph is called *Fan,* also known as *shell,* and is denoted by F_n. Let $u_1,\ u_2,\ \cdots, u_n$ be the vertices of C_n and let u_1 be the vertex that is connected to all other vertices, as illustrated in Figure 2.

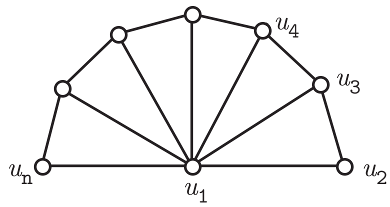

Figure 2. The fan F_n $(n = 8)$.

The degree set of F_n is $\{2, 3, n-1\}$, hence it is not 2-magic. For $n = 3$, we notice that $F_3 \equiv C_3$, which is totally magic and $IM(F_3) = \mathbb{N}$. Also, a typical magic labeling of $F_4 \cong K_4 - e$ is illustrated in Figure 3, for which we require that $a + b - 2z = a + b$ or $2z \equiv 0 \pmod{h}$; that is, h has to be even. On the other hand, if $h = 2r$, then F_4 is 4-magic (Figure 3). Therefore, $IM(F_4) = 2\mathbb{N} - \{2\}$.

From now on we will assume that $n \geq 5$.

Lemma 3.1. *If $n = 2k + 1 \geq 5$, then* $IM(F_n) = \begin{cases} \mathbb{N} - \{2, 3\} & \text{if } n = 7; \\ \mathbb{N} - \{2\} & \text{otherwise.} \end{cases}$

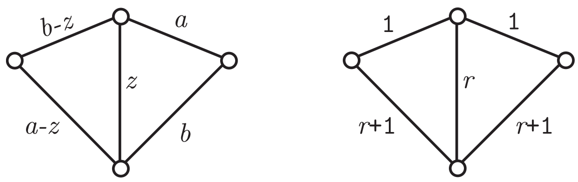

FIGURE 3. A typical magic labeling of F_4.

Proof. First we observe that for any $h > k$, the fan F_{2k+1} is h-magic. To show this, we consider the following labels $l(u_1u_2) = l(u_1u_n) = 1 - k,\ l(u_1u_i) = 1$ for all $1 < i < n$, and the remaining cycle edges by $k - 1,\ -k$, alternatively, with $l(u_2u_3) = 1 - k$. This is a magic labeling with the sum $l^+(u_j) = 0$. If $h = k$, then label the two edges u_1u_2 and u_1u_n by 3 and all other edges by 1. This is also a magic labeling with sum 4. Now, we may assume that $h < k$ and label the edges of F_n as illustrated in Figure 4.

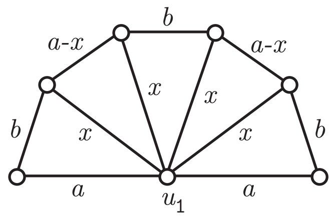

FIGURE 4. A typical magic labeling of F_7.

For this labeling we require $(2k - 2)x + 2a \equiv a + b \pmod h$ or

$$2(k-1)x \equiv b - a \pmod h. \tag{3.1}$$

If $\gcd(h, k - 1) = \delta > 1$, we choose $x = h/\delta,\ a = b \neq x$, which would provide a magic labeling in $\mathbb{Z}_h$ with sum $2a$; because, $l^+(u_1) = 2(k-1)h/\delta + 2a \equiv 2a \pmod h$. Finally, if $\gcd(h, k - 1) = 1$, we choose x to be the nonzero solution of the equation $(k-1)x \equiv 1 \pmod h$, $a \neq x$, and $b = a+2$. This provides a magic labeling of F_n with sum $2a + 2$ provided $h \neq 3$. □

Lemma 3.2. *If* $n = 2k \geq 6$, *then* $IM(F_n) = \begin{cases} \mathbb{N} - \{2\} & \text{if } 3|(k-1); \\ \mathbb{N} - \{2, 3\} & \text{otherwise.} \end{cases}$

Proof. First we try the labeling illustrated in Figure 5, for which we require $2(k-1)x \equiv 0 \pmod h$. If $h = 2r \geq 4$ is an even number, we choose $x = r$. This provides a magic labeling with sum $r + 2$.

If $\gcd(h, k - 1) = \delta > 1$, we choose $x = h/\delta$, which would provide a magic labeling in $\mathbb{Z}_h$ with sum $l^+(u_1) = 2 + 2x + (2k - 3)h/\delta \equiv 2 + x + 2(k-1)h/\delta \equiv 2 + x \pmod h$.

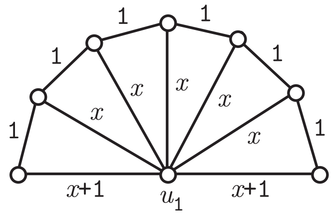

FIGURE 5. One possible magic labeling of F_8.

Now, we may assume that h is odd and $\gcd(h, k-1) = 1$. Consider the labeling illustrated in Figure 6.

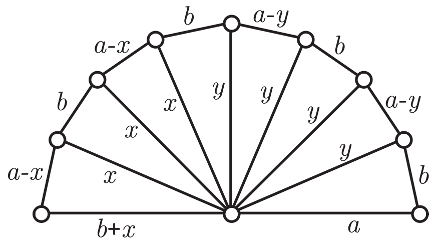

FIGURE 6. A typical magic labeling of F_{10}.

We have the option of labeling none, two, or four consecutive chords by y, which will produce the equations $(k-1)x \equiv 0 \pmod h$, $y + (k-2)x \equiv 0 \pmod h$, and $2y + (k-3)x \equiv 0 \pmod h$, respectively. Since $\gcd(k-2, k-3) = 1$, then $h \nmid (k-2)$ or $h \nmid (k-3)$.

If $h \nmid (k-2)$, then let $\gcd(h, k-2) = \delta$. Choose $y = -\delta$ and let x be the nonzero solution of the equation $(k-2)x \equiv \delta \pmod h$, and $a \neq x, y$. This would be a magic labeling of F_{2k} with sum $2a$.

If $h \nmid (k-3)$, then let $\gcd(h, k-3) = \delta$. Choose $y = -2\delta$ and let x be the nonzero solution of the equation $(k-3)x \equiv 2\delta \pmod h$, and $a \neq x, y$. This would be a magic labeling of F_{2k} with sum $2a$. Finally, since $\mathbb{Z}_3$ has only two nonzero elements, we only have the option $x = y$ and the equation $(k-1)x \equiv 0 \pmod 3$, which would be true if and only if $3|(k-1)$. □

4. CYCLES WITH A P_k CHORD

A *cycles with a P_k chord* is a cycle with the path P_k joining two nonconsecutive vertices of the cycle. Since the degree set of these graphs is $\{2, 3\}$, they are not 2-magic. Based on Observation 1.6, it is enough to consider the cases when $k = 2, 3$. The chord P_k splits C_n into two subcycles.

Depending on the number of edges of these subcycles, we will have different results. The next lemma is about cycles with a P_2 chord:

Lemma 4.1. *Let $G_{n,2}$ be the cycle C_n with a P_2 chord. Then*

$$IM(G_{n,2}) = \begin{cases} \mathbb{N} - \{2\} & \textit{both subcycles are even;} \\ 2\mathbb{N} - \{2\} & \textit{both subcycles are odd;} \\ \mathbb{N} - \{2,3\} & \textit{otherwise.} \end{cases}$$

Proof. Based on the observation 1.6, it is enough to consider C_3 and C_4 as the two subcycles.

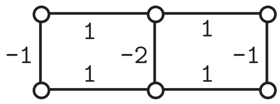

FIGURE 7. $G_{n,2}$ consists of two even subcycles.

Case I. Both subcycles are even. The labeling illustrated in Figure 7, proves that $G_{n,2}$ is h-magic for all $h \geq 3$.

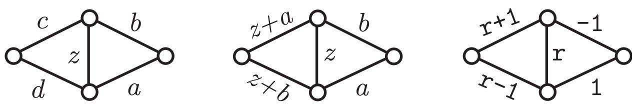

FIGURE 8. $G_{n,2}$ consists of two odd subcycles.

Case II. Both subcycles are odd. The typical labeling of $G_{n,2}$ in $\mathbb{Z}_h$ is illustrated in Figure 8. The requirement $a+z+d = c+d$ and $b+z+c = c+d$ imply that $c = a+z$ and $d = b+z$. Also, $a+b = c+d$ will result to $2z \equiv 0 \pmod{h}$ or $2|h$. On the other hand, if $h = 2r$, then the choice of $z = r,\ a = 1$, and $b = -1$ provides a zero sum result. Therefore, $IM(G_{n,2}) = 2\mathbb{N} - \{2\}$.

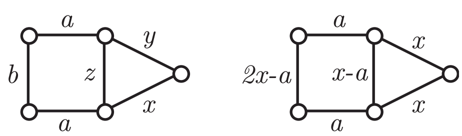

FIGURE 9. $G_{n,2}$ consists of one odd and one even subcycles.

Case III. Subcycles have different parities. The typical labeling of $G_{n,2}$ in $\mathbb{Z}_h$ is illustrated in Figure 9. The condition $a+x+z = a+y+z$ implies $x = y$. Also, the requirements $a+z+x = 2x$ will result to $z = x-a$ and $b = 2x-a$. Therefore, given $x \in \mathbb{Z}_h - \{0\}$, we need another nonzero element $a \neq x$, $2x$, hence $h \geq 4$. Therefore, the integer-magic spectrum of such graphs would be $\mathbb{N} - \{2, 3\}$. □

Corollary 4.2. *Let $G_{n,2k}$ be the cycle C_n with a P_{2k} chord. Then*

$$IM(G_{n,2k}) = \begin{cases} \mathbb{N} - \{2\} & \textit{both subcycles are even;} \\ 2\mathbb{N} - \{2\} & \textit{both subcycles are odd;} \\ \mathbb{N} - \{2,3\} & \textit{otherwise.} \end{cases}$$

Lemma 4.3. *Let $G_{n,3}$ be the cycle C_n with a P_3 chord. Then*

$$IM(G_{n,3}) = \begin{cases} \mathbb{N} - \{2\} & \textit{both subcycles are even;} \\ \mathbb{N} - \{2,3\} & \textit{both subcycles are odd;} \\ 2\mathbb{N} - \{2\} & \textit{otherwise.} \end{cases}$$

Proof. Based on the observation 1.6, it is enough to consider C_4 and C_5 as the two subcycles.

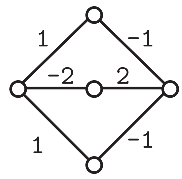

FIGURE 10. $G_{n,3}$ consists of two even subcycles.

Case I. Both subcycles are even (C_4). The labeling illustrated in Figure 10, shows that $G_{n,3}$ is h-magic for all $h \geq 3$; that is, $IM(G_{n,3}) = \mathbb{N} - \{2\}$.

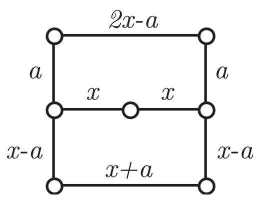

FIGURE 11. $G_{n,3}$ consists of two odd subcycles.

Case II. Both subcycles are odd (C_5). The typical magic labeling of $G_{n,3}$ in $\mathbb{Z}_h$ is illustrated in Figure 11, which has sum $2x$. Here, given $x \in \mathbb{Z}_h$, one needs another nonzero element $a \neq x, -x$. Hence, the graph cannot be 3-magic, and its integer-magic spectrum is $\mathbb{N} - \{2, 3\}$.

Case III. Subcycles have different parities. The typical magic labeling of $G_{n,3}$ in $\mathbb{Z}_h$ is illustrated in Figure 12. For the graph to be h-magic, we need $3a + c + x = a + c + x$ or $2a \equiv 0 \pmod{h}$; that is, h is even and the integer-magic spectrum of the graph would be $2\mathbb{N} - \{2\}$. □

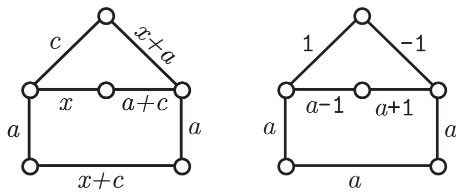

FIGURE 12. $G_{n,3}$ consists of one odd and one even subcycles.

Corollary 4.4. *Let $G_{n,2k+1}$ be the cycle C_n with a P_{2k+1} chord. Then*

$$IM(G_{n,2k+1}) = \begin{cases} \mathbb{N} - \{2\} & \textit{both subcycles are even;} \\ \mathbb{N} - \{2,3\} & \textit{both subcycles are odd;} \\ 2\mathbb{N} - \{2\} & \textit{otherwise.} \end{cases}$$

5. n-GON BOOKS

When k copies of C_n share a common edge, it will form an *n-gon book of k pages* and is denoted by $B(n,k)$. In this section, the integer-magic spectrum of $B(n,k)$ will be determined. Again, using the observation 1.6, we will only consider the two cases $n = 3, 4$.

Theorem 5.1. $IM(B(2n,k)) = \mathbb{N} - \{1 + (-1)^k\}$. *Also,*

$$IM(B(2n+1,k)) = \begin{cases} \mathbb{N} - \{d > 1 : d|(k-2)\} & k \textit{ is odd ;} \\ \mathbb{N} - \{d > 1 : d = 2, \textit{ or } d \textit{ is odd and } d|(k-2)\} & k \textit{ is even.} \end{cases}$$

Proof. As before, depending on whether n is even or odd it will be enough to consider C_4 and C_3, respectively. If n is even and k is odd, we will label the common edge by -1 and top edges $1,\ -1$ alternatively. This provides a magic labeling with sum 0.

If n and k are both even, we will label the common edge by -1 and one top edge by 2 the remaining top edges $-1,\ 1$ alternatively. This will produce a zero sum.

Suppose n is odd. We will label the common edge by z and the edges of the i^{th} cycle by $a_i,\ b_i$, as illustrated in Figure 13. Since the sum of the labels of the edges at the two vertices of the common edge should be the same, we will end up with the following set of equations:

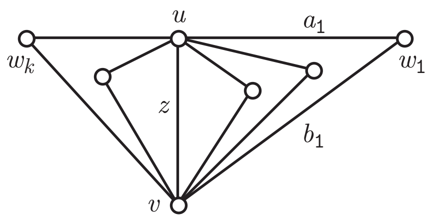

FIGURE 13. A typical labeling of $B(3, k)$.

$$a_i + b_i \equiv a_1 + b_1 \pmod{h}; \tag{5.1}$$

$$z + \sum_{i=1}^{k} a_i \equiv a_1 + b_1 \pmod{h}; \tag{5.2}$$

$$z + \sum_{i=1}^{k} b_i \equiv a_1 + b_1 \pmod{h}. \tag{5.3}$$

From (5.2)and (5.3) we get $\sum(a_i - b_i) = 0$; one option would be $a_i = b_i = 1$ with the choice of $z = 2 - k$. Then $l^+(u) = l^+(v) \equiv 2 \pmod{h}$, provided $h \nmid (k-2)$. Therefore, for any $h > 2$, if $h \nmid (k-2)$, then $h \in IM(B(3, k))$.

Now suppose $h|(k-2)$. From (5.1) we get $2\sum a_i = k(a_1 + b_1) \equiv 2(a_1 + b_1) \pmod{h}$, which together with (5.2) and (5.3) imply $2z \equiv 0 \pmod{h}$. For odd values of h, this would be $z = 0$, not an acceptable label and the graph is not h-magic.

If $h = 2r$ is even, then $B(3, k)$ would be h-magic; the choices $z = r$, $a_i = r + 1$, and $b_i = 1$ will provide a magic labeling of the graph in $\mathbb{Z}_h$ with the sum $l^+(u) = k(r+1) + r \equiv r + 2 \pmod{h}$.

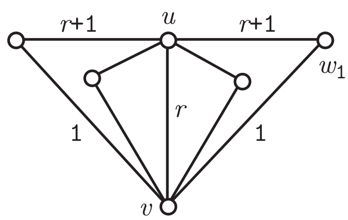

FIGURE 14. A magic labeling of $B(3, k)$, when $h = 2r$ and $h|(k-2)$.

Finally we observe that when k is even, the degrees of vertices do not have the same parity and the book is not 2-magic. □

6. Suggestions for Further Research

Note that wheels and fans can be described by the join operation $W_n = C_n + v$ and $F_n = P_{n-1} + v$, even the triangular book $B(3,k)$ can be described by $ST(k) + v$, where $ST(k) = K(1,k)$ is the star with k leaves. It would be natural to establish a relationship between $IM(G)$ and $IM(G+v)$.

Problem 6.1. *Given two graphs G_1 and G_2, find a relationship between $IM(G_1)$, $IM(G_2)$ and $IM(G_1 + G_2)$.*

Also, note that $IM(G) \subset IM(G \times K_2)$. Here, equality may not occur; for example, if $G = P_3$, then $IM(P_3) = \emptyset$, while $IM(P_3 \times P_2) = \mathbb{N} - \{2\}$. However, if G is a regular graph, then $IM(G) = IM(G \times P_2) = \cdots = IM(G \times Q_n)$, where Q_n is the hypercube (or n-cube).

Problem 6.2. *Characterize the graphs for which $IM(G) = IM(G \times P_2)$.*

Clearly, if the graphs G and H are h-magic, then so is $G \times H$, their Cartesian products, which implies that $IM(G) \cap IM(H) \subset IM(G \times H)$. Here, again the equality may not occur, as can be verified for $G = P_3$, which leads us to the following problem:

Problem 6.3. *Characterize the graphs for which $IM(G) \cap IM(H) = IM(G \times H)$.*

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