

# INTEGER-MAGIC SPECTRA OF AMALGAMATIONS OF STARS AND CYCLES

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ABSTRACT. For any positive integer  $k$ , a graph  $G = (V, E)$  is said to be  $\mathbb{Z}_k$ -magic if there exists a labeling  $l : E(G) \rightarrow \mathbb{Z}_k - \{0\}$  such that the induced vertex set labeling  $l^+ : V(G) \rightarrow \mathbb{Z}_k$  defined by

$$l^+(v) = \sum \{ l(uv) : uv \in E(G) \}$$

is a constant map. For a given graph  $G$ , the set of all  $h \in \mathbb{Z}_+$  for which  $G$  is  $\mathbb{Z}_h$ -magic is called the integer-magic spectrum of  $G$  and is denoted by  $IM(G)$ . In this paper, we will determine the integer-magic spectra of the graphs which are formed by the amalgamation of stars and cycles. In particular, we will provide examples of graphs that for a given  $n > 2$ , they are not  $h$ -magic for all values of  $2 \leq h \leq n$ .

## 1. INTRODUCTION

For any abelian group  $A$ , written additively, any mapping  $l : E(G) \rightarrow A - \{0\}$  is called a *labeling*. Given a labeling on the edge set of  $G$  one can introduce a vertex set labeling  $l^+ : V(G) \rightarrow A$  as follows:

$$l^+(v) = \sum \{ l(uv) : uv \in E(G) \}.$$

A graph  $G$  is said to be *A-magic* if there is a labeling  $l : E(G) \rightarrow A - \{0\}$  such that for each vertex  $v$ , the sum of the labels of the edges incident with  $v$  are all equal to the same constant; that is,  $l^+(v) = c$  for some fixed  $c \in A$ . We will call  $\langle G, l \rangle$  an *A-magic graph* with sum  $c$ . In general, a graph  $G$  may admit more than one labeling to become an *A-magic graph*; for

example, if  $|A| > 2$  and  $l : E(G) \rightarrow A - \{0\}$  is a magical labeling of  $G$  with sum  $c$ , then  $\lambda : E(G) \rightarrow A - \{0\}$ , the inverse labeling of  $l$ , defined by  $\lambda(uv) = -l(uv)$  will provide another magical labeling of  $G$  with sum  $-c$ .

The original concept of  $A$ -magic graph is due to J. Sedlacek [18, 19], who defined it to be a graph with a real-valued edge labeling such that

- (1) distinct edges have distinct nonnegative labels; and
- (2) the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Given a graph  $G$ , the problem of deciding whether  $G$  admits a magic labeling is equivalent to the problem of deciding whether a set of linear homogeneous Diophantine equations has a solution [20]. At present, given an abelian group, no general efficient algorithm is known for finding magic labelings for general graphs.

When  $A = \mathbb{Z}$ , the  $\mathbb{Z}$ -magic graphs were considered in Stanley [20, 21], he pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. When the group is  $\mathbb{Z}_k$ , we shall refer to the  $\mathbb{Z}_k$ -magic graph as  $k$ -magic. Graphs which are  $k$ -magic had been studied in [2, 5, 6, 9, 10, 12, 14]. For convenience, we will use the notation 1-magic instead of  $\mathbb{Z}$ -magic. Doob [2, 3, 4], also considered  $A$ -magic graphs where  $A$  is an abelian group. He determined which wheels are  $\mathbb{Z}$ -magic.

A graph  $G = (V, E)$  is called *fully magic* [9, 11, 17, 22, 25, 26] if it is  $A$ -magic for every abelian group  $A$ , and it is called *non-magic* if for every abelian group  $A$  it is not  $A$ -magic. Also, a graph  $G$  is said to be  $\mathbb{N}$ -magic if there exists a labeling  $l : E(G) \rightarrow \mathbb{N}$  such that  $l^+(v)$  is a constant, for every  $v \in V(G)$ . It is well-known that a graph  $G$  is  $\mathbb{N}$ -magic if and only if each edge of  $G$  is contained in a 1-factor (a perfect matching) or a  $\{1, 2\}$ -factor [11, 17, 26]. Berge [1] called a graph *regularisable* if a regular multigraph could be obtained from  $G$  by adding edges parallel to the edges of  $G$ . In fact, a graph is regularisable if and only if it is  $\mathbb{N}$ -magic. For  $\mathbb{N}$ -magic graphs, readers are referred to [7, 8, 9, 10, 12, 22, 23, 24]. The notion of  $\mathbb{Z}$ -magic is weaker than  $\mathbb{N}$ -magic. Figure 1 shows a graph which is  $\mathbb{Z}$ -magic but not  $\mathbb{N}$ -magic.

**Observation 1.1.** *For any  $n \geq 3$ , the path of order  $n$  is non-magic.*

**Observation 1.2.** *In any magical labeling of a cycle the edges should alternatively be labeled the same group elements.*

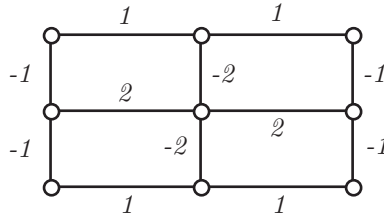


FIGURE 1. The graph  $P_3 \times P_3$  is  $\mathbb{Z}$ -magic but is not  $\mathbb{N}$ -magic.

*Proof.* Let  $l : E(C_n) \rightarrow A$  be a magic labeling and  $e_1, e_2, e_3, e_4$  be four consecutive edges of  $C_n$ . Then  $l(e_1) + l(e_2) = l(e_2) + l(e_3)$  and  $l(e_2) + l(e_3) = l(e_3) + l(e_4)$ . Which implies that  $l(e_1) = l(e_3)$  and  $l(e_2) = l(e_4)$ .  $\square$

**Observation 1.3.**  $C_{2n}$ , the cycle of order  $2n$ , with a pendant edge is non-magic.

*Proof.* By the Observation 1.2, it is enough to prove the statement for  $C_4$ . As illustrated in the Figure 2, in any labeling, the sum of labels of the edges incident with vertex  $v$  needs to be equal to the sum of labels of the edges incident with  $u$ ; that is,  $x + y + z = x + y \implies z = 0$ , a contradiction.  $\square$

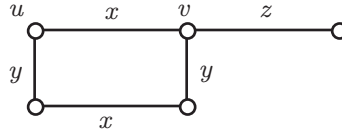


FIGURE 2. The cycle  $C_4$  with an edge pendant is non-magic.

In this paper, we will denote the set of positive integers by  $\mathbb{N}$ , and for any  $k > 0$ ,

$$k\mathbb{N} = \{ kn : n \in \mathbb{N} \}, \text{ also } k + \mathbb{N} = \{ k + n : n \in \mathbb{N} \}.$$

For a given graph  $G$  the set of all positive integers  $h$  for which  $G$  is  $\mathbb{Z}_h$ -magic (or simply  $h$ -magic) is called the *integer-magic spectrum* of  $G$  and is denoted by  $IM(G)$ . Since any regular graph is fully magic, then it is  $h$ -magic for all positive integers  $h \geq 2$ ; therefore,  $IM(G) = \mathbb{N}$ . For more general results on integer-magic spectrum of graphs, the reader is referred to [13, 15, 16].

A graph  $G$  with a fixed vertex  $u$  will be denoted by the ordered pair  $(G, u)$ . Given two ordered pairs  $(G, u)$  and  $(H, v)$ , one can form a new graph by amalgamation: form the disjoint union of  $G$  and  $H$  and identify  $u$  and  $v$ . The resulting graph will be denoted by  $(G, u) \circ (H, v)$ .

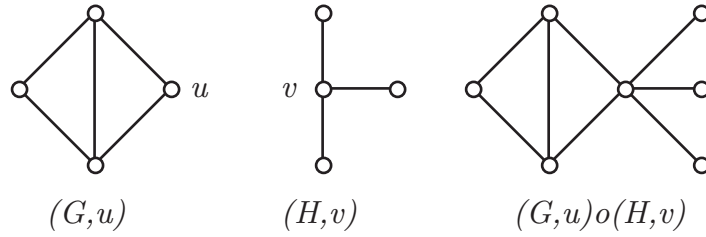


FIGURE 3. Amalgamation Construction.

For convenience the complete bipartite graph  $K(1, m)$ , known as star with  $m$  leaves, will be denoted by  $ST(m)$ . Given a star  $ST(m)$  and a cycle  $C_n$ , depending on whether we identify the center of the star with a vertex of  $C_n$  or identify an end-vertex of the star with a vertex of  $C_n$ , the amalgamation of these two graphs will result in two non-isomorphic graphs. We will denote the first one by  $ST(m)\#C_n$ , and the latter by  $ST(m)\@C_n$ . Figure 4 illustrates the two different amalgamations of  $ST(3)$  and  $C_4$ .

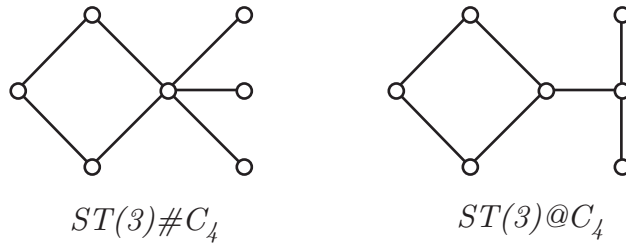


FIGURE 4. Two different amalgamations of  $ST(3)$  and  $C_4$ .

## 2. INTEGER-MAGIC SPECTRUM OF $ST(m)\#C_n$ .

By the Observation 1.2, in any magic labeling of  $C_n$ , labels of the edges alternates. One immediate consequence of this fact is that, the graph  $ST(m)\#C_3$  (or  $ST(m)\#C_4$ ) is  $h$ -magic if and only if the graph  $ST(m)\#C_{2k+1}$  (or  $ST(m)\#C_{2k}$ ) is  $h$ -magic. As a result, we will only concentrate on cycles  $C_3$  and  $C_4$ . Also, we observe that since  $\{1, 2\}$  is a subset of the degree set of these types of graphs,  $ST(m)\#C_n$  can not be 2-magic.

**Theorem 2.1.**  $IM(ST(1)\#C_4) = \emptyset$ .

*Proof.* This is a direct result of the Observation 1.3. □

**Theorem 2.2.**  $IM(ST(2)\#C_4) = 2 + 2\mathbb{N}$ .

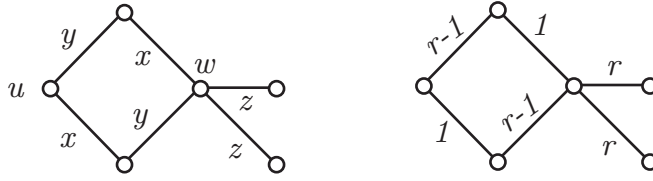


FIGURE 5. General labeling of  $ST(m)\#C_4$ . Here  $z = x + y$ .

*Proof.* We observe that  $ST(2)\#C_4$  is  $h$ -magic if and only if  $2|h$  and  $h > 2$ . Because, as illustrated in the Figure 5, we need to have  $l^+(w) = l^+(u)$  or  $2z \equiv 0 \pmod{h}$ , and this means  $z$  is an order two element of the group  $\mathbb{Z}_h$ . Therefore,  $2|h$ . On the other hand, if  $h = 2r$ , then the selection of  $x = 1$ ,  $y = r - 1$ ,  $z = r$  will work.  $\square$

**Theorem 2.3.** *For any positive integer  $m \geq 2$ , the graph  $ST(m)\#C_4$  is  $h$ -magic if and only if  $h > 2$  and  $\gcd(m, h) > 1$ . Therefore,*

$$IM(ST(m)\#C_4) = (2 + \mathbb{N}) - \{ h \in \mathbb{N} : \gcd(m, h) = 1 \}.$$

*Proof.* In  $ST(m)\#C_4$  there are  $m$  pendant edges attached to the vertex  $w$  and the equation  $l^+(w) = l^+(u)$  implies that

$$(2.1) \quad mz \equiv 0 \pmod{h}.$$

If  $\gcd(m, h) = 1$ , then the equation 2.1 will be equivalent to  $z \equiv 0 \pmod{h}$ , which does not provided a non-zero solution. Now suppose  $\gcd(m, h) = \delta > 1$ . If  $\delta = h$ , then we use  $x = y = 1$  and  $z = 2$  as our labels, with  $l^+ \equiv 2$ . Suppose  $1 < \delta < h$ , and let  $h = \delta r$  ( $r \geq 2$ ). Then the selection  $x = 1$ ,  $y = r - 1$ , and naturally  $z = r$  will work with  $l^+ \equiv r$ .  $\square$

**Corollary 2.4.** *If  $p$  is a prime number bigger than 2, then*

$$IM(ST(p)\#C_4) = p\mathbb{N}.$$

**Corollary 2.5.** *If  $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$  is an odd positive integer, then*

$$IM(ST(m)\#C_4) = \bigcup_{i=1}^k (p_i \mathbb{N}).$$

*If  $m = 2^\beta p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$  is even, then*

$$IM(ST(m)\#C_4) = (2 + 2\mathbb{N}) \bigcup_{i=1}^k (p_i \mathbb{N}).$$

**Theorem 2.6.**  $IM(ST(1)\#C_3) = 2 + 2\mathbb{N}$ .

*Proof.* For graph  $ST(1)\#C_3$  to be  $h$ -magic, as illustrated in the Figure 6, we need to have  $2x+z = z$  or  $2x \equiv 0 \pmod{h}$ . That is,  $x$  is a non-zero element of  $\mathbb{Z}_h$  with order 2. Therefore,  $2|h$ . On the other hand, if  $h = 2r$ , then the selections  $x = r$ ,  $y = 1$  will work; that is, the graph  $ST(1)\#C_3$  has  $h$ -magic.  $\square$

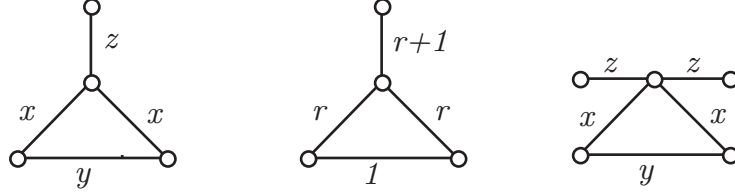


FIGURE 6.  $ST(1)\#C_3$  and  $ST(2)\#C_3$ .

**Theorem 2.7.**  $IM(ST(2)\#C_3) = \mathbb{N} - \{2, 3\}$ .

*Proof.* For the graph  $ST(2)\#C_3$  to be  $h$ -magic, as illustrated in the Figure 6, we need to have  $2x + 2z = z$  or  $2x + z \equiv 0 \pmod{h}$ . Here  $z = x + y$ , therefore we need  $3x + y \equiv 0 \pmod{h}$ , and this equation does not provide non-zero solutions for  $h = 3$ . If  $h > 3$ , then the selections  $x = 1$ ,  $y = h - 3$  will work. This means the graph is  $h$ -magic for all  $h \geq 4$ . Furthermore,  $ST(2)\#C_3$  is  $\mathbb{Z}$ -magic; because, the labels  $x = -1$ ,  $y = 3$  will work with  $l^+ \equiv 2$ .  $\square$

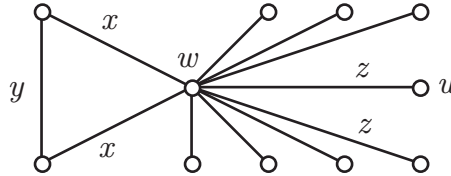


FIGURE 7. A typical labeling of  $ST(m)\#C_3$ . Here,  $z = x + y$ .

Note that when  $m > 1$ , the graph  $ST(m)\#C_3$  is  $\mathbb{Z}$ -magic; because, the labels  $x = 1-m$ ,  $y = 1+m$ , and  $z = 2$  will work with  $l^+ \equiv 2$ . In addition, any magical labeling of  $ST(m)\#C_3$ , as illustrated in the Figure 7, uses three non-zero group elements  $x, y$ , and  $z = x + y$ . In particular, we need to have  $l^+(u) = l^+(w)$ , which implies

$$(2.2) \quad (m-1)z + 2x \equiv 0 \pmod{h},$$

or equivalently

$$(2.3) \quad (m+1)x + (m-1)y \equiv 0 \pmod{h}.$$

**Theorem 2.8.** *The graph  $ST(m)\#C_3$  is 3-magic if and only if  $3|m$ .*

*Proof.* If  $m = 3k$ , the labeling  $x = y = 1$  will work with  $l^+ \equiv 2 \pmod{3}$ . If  $m = 3k + 1$ , the equation 2.3 will become  $(3k + 2)x + 3ky \equiv 0 \pmod{3}$ , which is equivalent to  $2x \equiv 0 \pmod{3}$ , or  $x \equiv 0$ , which is not an acceptable answer. If  $m = 3k + 2$ , the equation 2.3 will become  $(3k + 3)x + (3k + 1)y \equiv 0 \pmod{3}$ , which is equivalent to  $y \equiv 0$ , not an acceptable answer.  $\square$

**Theorem 2.9.** *For every  $h > m + 1$ , the graph  $ST(m)\#C_3$  is  $h$ -magic.*

*Proof.* If  $h > m + 1$ , then we choose  $x = h - (m - 1)$ ,  $y = m + 1$ , and naturally  $z = 2$ . We notice that  $x, y, z$  are non-zero elements of  $\mathbb{Z}_h$  and  $l^+(w) = mz + 2x = 2m + 2(h - m + 1) = 2h + 2 \equiv 2 \pmod{h}$ .  $\square$

**Theorem 2.10.** *If  $m > 2$ , then the graph  $ST(m)\#C_3$  is  $m$ -magic.*

*Proof.* The labeling  $x = y = 1$  works with  $l^+ \equiv 2$ .  $\square$

**Theorem 2.11.** *The graph  $ST(m)\#C_3$  is  $h$ -magic for all even positive integers  $h \geq 4$ .*

*Proof.* To prove the theorem it is enough to show that the equation 2.2 has two distinct non-zero solutions for  $z$  and  $x$  in  $\mathbb{Z}_h$ . Let  $h = 2^\alpha \mu$ , where  $\mu$  is an odd number, and consider the equations

$$(2.4) \quad (m - 1)z + 2x \equiv 0 \pmod{2^\alpha},$$

$$(2.5) \quad (m - 1)z + 2x \equiv 0 \pmod{\mu}.$$

Given any  $z$ , the equation 2.5 has the solution  $x \equiv -\bar{2}(m - 1)z$  for  $x$ , where  $\bar{2}$  is the multiplicative inverse of 2 in  $\mathbb{Z}_\mu$ . For the equation 2.4 we consider two cases:

Case 1. When  $m = 2k + 1$  is odd, then the equation 2.4 becomes  $2kz + 2x \equiv 0 \pmod{2^\alpha}$  or  $kz + x \equiv 0 \pmod{2^{\alpha-1}}$ , and for any  $z$  we will have  $x \equiv -kz \pmod{2^{\alpha-1}}$ . Therefore,  $x \equiv -kz \pmod{2^{\alpha-1} \cdot \mu}$ , and for any non-zero  $z \in \mathbb{Z}_h$ , one can choose either  $x \equiv -kz$  or  $x \equiv -kz + 2^{\alpha-1} \cdot \mu \pmod{h}$  for the solution of the equation 2.2.

Case 2. When  $m = 2k$  is even, then  $z$  has to be even. Let  $z = 2\xi$ . Equation 2.4 becomes  $(m - 1)\xi + x \equiv 0 \pmod{2^{\alpha-1}}$ , and for any  $\xi$  we will have  $x \equiv -(m - 1)\xi \pmod{2^{\alpha-1}}$ . Therefore,  $x \equiv -(m - 1)\xi \pmod{2^{\alpha-1} \cdot \mu}$ , and for any non-zero  $z = 2\xi \in \mathbb{Z}_h$ , one can choose either  $x \equiv -(m - 1)\xi$  or  $x \equiv -(m - 1)\xi + 2^{\alpha-1} \cdot \mu \pmod{2^\alpha}$  for the solution of the equation 2.2.

□

As an application of the Theorem 2.11, we will show that  $ST(m)\#C_3$  is always 4-magic. Using the notation and the process of the theorem, we will consider two cases:

Case 1. If  $m = 2k + 1$ , then we need  $kz + x \equiv 0 \pmod{2}$  or  $x \equiv -kz \pmod{2}$ . For non-zero  $z = 3$ , we have two choices of either  $x \equiv -3k \pmod{4}$  or  $x \equiv -3k + 2 \pmod{4}$ , which translates to either  $x = 1$  or  $x = 2$ .

Case 2. If  $m = 2k$ , then we will deal with the equation  $(2k - 1)z + 2x \equiv 0 \pmod{4}$ , which implies that  $z = 2\xi$ . With this consideration the equation becomes  $(2k - 1)\xi + x \equiv 0 \pmod{2}$  or  $x \equiv \xi \pmod{2}$ . Now for the only choice of  $z = 2$  (or  $\xi = 1$ ) we will have two choices of either  $x \equiv 1$  or  $x \equiv 3$  and the corresponding values of  $y$  will be 1 or 3, respectively.

**Theorem 2.12.** *Let  $h \geq 3$  be an odd positive integer. Then  $ST(m)\#C_3$  is  $h$ -magic if and only if  $h$  is not a divisor of  $m + 1$  and  $m - 1$ .*

*Proof.* Suppose  $h$  is a divisor of  $m + 1$  or  $m - 1$ . Since  $\gcd(m + 1, m - 1) = 1$  or  $2$  and  $h$  is odd, then  $h$  can only be divisor of one of them. Without loss of generality, we may assume that  $h$  is an odd divisor of  $m + 1$ . As a result  $\gcd(h, m - 1) = 1$ , and equation 2.3,  $(m + 1)x + (m - 1)y \equiv 0 \pmod{h}$ , becomes  $y \equiv 0 \pmod{h}$ , that does not provide non-zero solution for  $y$ .

Conversely, assume that  $h$  is not a divisor of  $m - 1$  and  $m + 1$ , and let  $m = hq + r$ . Then  $r \neq \pm 1$ ; otherwise, one of  $m - 1$  or  $m + 1$  will be divisible by  $h$ . As a result, the selections of  $x = d + 1 - r$  and  $y = 1 + r$  are valid and will work with  $l^+(w) = mz + 2x = 2(hq + r) + 2(d + 1 - r) \equiv 2 \pmod{h}$ . □

We conclude the section by the following theorem, which is the natural consequence of the Theorems 2.7 through 2.12. This theorem will completely determine the integer-magic spectrum of  $ST(m)\#C_3$ .

**Theorem 2.13.** *If  $m \geq 2$ , then the integer-magic spectrum of the graph  $ST(m)\#C_3$  is*

$$IN - \{ d \in IN : d = 2 \text{ or } d \text{ is an odd divisor of } m + 1 \text{ or } m - 1 \}.$$

As an application of this theorem, consider the graph  $ST(134)\#C_{23}$ . To find its integer-magic spectrum it is enough to consider  $ST(134)\#C_3$ , where  $m = 134$ . Now the odd divisors of  $m + 1 = 135$  and  $m - 1 = 133$  are 3, 5, 7, 9, 15, 19, 27, 45, 133, and 135. Therefore,

$$IM(ST(134)\#C_{23}) = IN - \{ 2, 3, 5, 7, 9, 15, 19, 27, 45, 133, 135 \}.$$

### 3. INTEGER-MAGIC SPECTRUM OF $ST(m)@C_n$ .

By the Observation 1.2, in any magic labeling of  $C_n$ , the labels of the edges alternates. One immediate consequence of this fact is that, the graph  $ST(m)@C_3$  (or  $ST(m)@C_4$ ) is  $h$ -magic if and only if the graph  $ST(m)@C_{2k+1}$  (or  $ST(m)@C_{2k}$ ) is  $h$ -magic. As a result, we will only concentrate on cycles  $C_3$  and  $C_4$ . Also, we observe that since  $\{1, 2\}$  is a subset of the degree set of these types of graphs, they can not be 2-magic. In general any magic labeling of  $ST(m)@C_3$ , as illustrated in the Figure 8, uses three non-zero distinct group elements  $x, y$ , and  $z = x + y$ . Therefore, for any abelian group  $A$ , a necessary condition for  $ST(m)@C_3$  to be  $A$ -magic is that  $|A| \geq 4$ . Hence, for any  $m \in \mathbb{N}$ , the graph  $ST(m)@C_3$  is not 3-magic. Moreover, when  $m \geq 4$ , then  $ST(m)@C_3$  is  $\mathbb{Z}$ -magic; because, the labels  $x = m - 1$ ,  $y = 3 - m$  work with  $l^+ \equiv 2$ . Therefore, the integer-magic spectrum of these graphs will be contained in  $\mathbb{N} - \{2, 3\}$ .

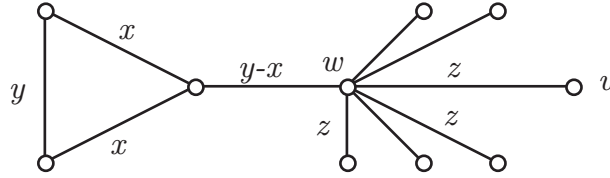


FIGURE 8. A typical magical labeling of  $ST(m)@C_3$ . Here  $z = x + y$ .

**Theorem 3.1.**  $IM(ST(m)@C_4) = \emptyset$  for every  $m \geq 1$ .

*Proof.* This is a direct result of the Observation 1.3. □

We observe that  $ST(1)@C_3 = ST(1)\#C_3$ , therefore  $IM(ST(1)@C_3) = 2 + 2\mathbb{N}$ . Also, by 1.1,  $ST(2)@C_3$  is non-magic, or  $IM(ST(2)@C_3) = \emptyset$ .

To determine the integer-magic spectrum of  $ST(m)@C_3$ , from now on we will assume that  $m \geq 3$ . Also, in any magic labeling of  $ST(m)@C_3$  one needs to have  $l^+(w) = l^+(u)$  or  $(m-1)z + y - x = z$ , which implies

$$(3.1) \quad (m-3)x + (m-1)y \equiv 0 \pmod{h},$$

or equivalently ( $z = x + y$ )

$$(3.2) \quad (m-1)z - 2x \equiv 0 \pmod{h},$$

**Theorem 3.2.**  $IM(ST(3)@C_3) = 2 + 2IN$ .

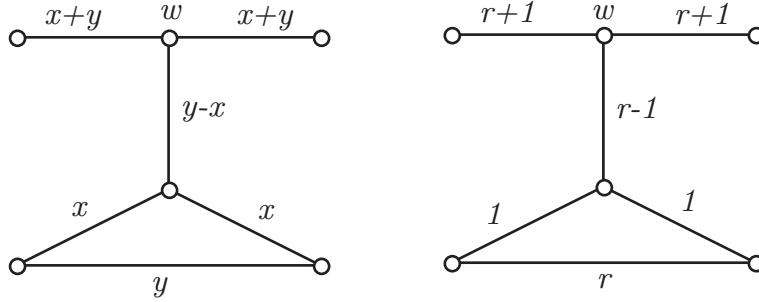


FIGURE 9.  $IM(ST(3)@C_3) = 2 + 2IN$ .

*Proof.* When  $m = 3$ , the equation 3.1 will become  $2y \equiv 0 \pmod{h}$ . This means that  $y$  is a member of  $\mathbb{Z}_h$ , which has order 2, therefore  $2|h$  and  $h$  is even. On the other hand, whenever  $h = 2r$ , the graph is  $h$ -magic, as illustrated in the Figure 9.  $\square$

**Theorem 3.3.** *If  $m \geq 4$ , then the graph  $ST(m)@C_3$  is  $\mathbb{Z}$ -magic.*

*Proof.* We observe that the choices of  $x = m - 1$ ,  $y = -m + 3$  and  $z = 2$  will give us three distinct non-zero integers with  $l^+(w) = (m - 1)z + y - x = 2(m - 1) - m + 3 - m + 1 = 2$ .  $\square$

**Theorem 3.4.** *For any abelian group  $A$ , if  $|A| \leq 4$ , then  $ST(4)@C_3$  is not  $A$ -magic. Furthermore,  $IM(ST(4)@C_3) = IN - \{2, 3, 4\}$ .*

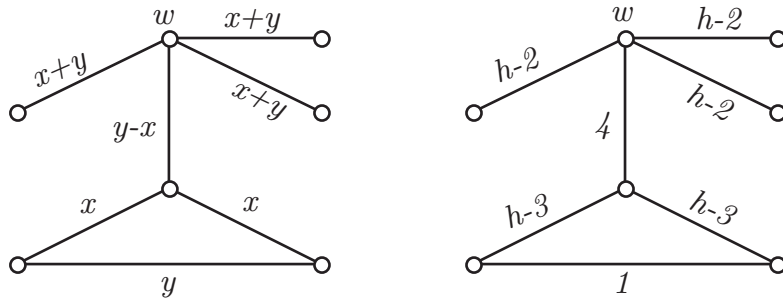


FIGURE 10.  $IM(ST(4)@C_3) = IN - \{2, 3, 4\}$ .

*Proof.* Any magic labeling of this graph, as illustrated by the Figure 10, will require three distinct non-zero elements of the abelian group  $A$ ; namely,  $x$ ,  $y$ , and  $x + y$ . Also, in this case, equation 3.1

becomes  $x + 3y = 0 \pmod{h}$ . This implies that  $x + 2y$  is another non-zero element of this group other than  $x, y$ , and  $x + y$ . Therefore, the group  $A$  must have at least five elements. Furthermore, as illustrated in the Figure 10, the graph is  $h$ -magic for every  $h \geq 5$ .  $\square$

**Corollary 3.5.** *The graph  $ST(m)@C_3$  is not 3-magic.*

**Theorem 3.6.** *For any  $m \geq 5$  the graph  $ST(m)@C_3$  is  $m$ -magic.*

*Proof.* The choices of  $x = m - 1$ ,  $y = 3$ , and  $z = 2$  will provide three distinct non-zero elements of  $\mathbb{Z}_m$  with  $l^+(w) = 2$ .  $\square$

**Theorem 3.7.** *If  $m$  is an odd positive integer, then the graph  $ST(2k + 1)@C_3$  is 4-magic.*

*Proof.* We consider two cases:

Case 1. If  $m = 4k + 1$ , then the choices of  $x = 2$ ,  $y = 3$ , and  $z = 1$  will work with  $l^+(w) = 4k + 1 \equiv 1 \pmod{4}$ .

Case 2. If  $m = 4k + 3$ , then the choices of  $x = 3$ ,  $y = 2$ , and  $z = 1$  will work with  $l^+(w) = (4k + 2) + 3 \equiv 1 \pmod{4}$ .  $\square$

**Theorem 3.8.** *If  $m = 2k + 1 \geq 5$ , then the integer-magic spectrum of  $ST(2k + 1)@C_3$  is*

$$\mathbb{N} - \{2\} \cup \{d > 1 : d \text{ is an odd divisor of one of } m - 1, m - 2, \text{ or } m - 3\}.$$

*Proof.* We will prove the theorem in five steps:

Step 1. In this part we will show that for any  $h \geq 2k$  the graph  $ST(2k + 1)@C_3$  is  $h$ -magic.

Because, the labeling of  $x = k$ ,  $y = h + 1 - k$ , and naturally,  $z = 1$  will work. Here we note that  $y - x = h + 1 - 2k \not\equiv 0 \pmod{h}$  and  $l^+(w) = 2kz + y - x \not\equiv 1 \pmod{h}$ .

Step 2. If  $h$  is any divisor of  $m - 2$ , then  $ST(m)@C_3$  is not  $h$ -magic. Because, the equation 3.2, is equivalent to  $(m - 2)z + y - x \equiv 0 \pmod{h}$ , and since  $h$  is a divisor of  $m - 2$ , we will have  $y - x \equiv 0 \pmod{h}$ , which is not an acceptable solution.

Step 3. If  $h$  is an odd divisor of either  $m - 1 = 2k$  or  $m - 3 = 2k - 2$ , then  $ST(2k + 1)@C_3$  is not  $h$ -magic. Because, the equation 3.1 becomes  $(2k - 2)x + 2ky \equiv 0 \pmod{h}$  or  $(k - 1)x + ky \equiv 0 \pmod{h}$ . Since  $\gcd(k - 1, k) = 1$ , without loss of generality, we may assume that  $d|(k - 1)$  and  $\gcd(d, k) = 1$ . As a result will get  $y \equiv 0 \pmod{h}$ , which is not an acceptable answer.

Step 4. If  $h$  is any odd number that is not a divisor of any one of  $m-1 = 2k$ ,  $m-2 = 2k-1$ ,  $m-3 = 2k-2$ , then  $ST(m)@C_3$  is  $h$ -magic. Because, the labels  $x \equiv k, y \equiv 1-k$ , and naturally  $z = 1$  are three non-zero distinct elements of  $\mathbb{Z}_h$ , will work with  $l^+ \equiv 1$ . Note that  $y - x \equiv 1 - 2k \not\equiv 0 \pmod{h}$ .

Step 5. If  $4 < h \leq m-1 = 2k$  is an even number, then  $ST(2k+1)@C_3$  is  $h$ -magic. Because, the equation 3.2 becomes  $2kz - 2x \equiv 0 \pmod{h}$ , which is equivalent to  $kz - x \equiv 0 \pmod{h/2}$ . Now for any non-zero  $z \in \mathbb{Z}_h$ , we have two choices for  $x$ ; namely,  $x \equiv kz \pmod{h}$ , or  $x \equiv kz + \frac{h}{2} \pmod{h}$ .

□

### Examples 3.9.

- (a)  $IM(ST(5)@C_3) = \mathbb{N} - \{2, 3\}$ . Here,  $m = 5$ . We need to exclude 2 and the odd divisors of  $m-1 = 4$ ,  $m-2 = 3$ , and  $m-3 = 2$ .
- (b)  $IM(ST(7)@C_3) = \mathbb{N} - \{2, 3, 5\}$ . Here,  $m = 7$ . We need to exclude 2 and the odd divisors of  $m-1 = 6$ ,  $m-2 = 5$ , and  $m-3 = 4$ .
- (c)  $IM(ST(45)@C_3) = \mathbb{N} - \{2, 3, 7, 11, 21, 43\}$ . Here,  $m = 45$ . We need to exclude 2 and the odd divisors of  $m-1 = 44$ ,  $m-2 = 43$ , and  $m-3 = 42$ .
- (d)  $IM(ST(135)@C_{95}) = \mathbb{N} - \{2, 3, 7, 11, 19, 33, 67, 133\}$ .

**Theorem 3.10.** *If  $m = 2k \geq 4$ , then the integer-magic spectrum of  $ST(m)@C_3$  is*

$$\mathbb{N} - \{ h > 1 : h|(2m-4), \text{ or } h|(m-1), \text{ or } h|(m-3) \}.$$

*Proof.* We will prove the theorem in three steps:

Step 1. If  $h > 2m-4$ , then  $ST(m)@C_3$  is  $h$ -magic. Because, the choices of  $x = m-1$ ,  $y = h-m+3$ , and naturally  $z = x + y = 2$ , will work with  $l^+(w) = 2(m-1) + 4 - 2m = 2$ . Also, note that  $y - x = 4 - 2m \not\equiv 0 \pmod{h}$ .

Step 2. If  $h > 1$  is any divisor of  $2m-4$ , then the graph  $ST(m)@C_3$  is not  $h$ -magic. It is enough to prove this statement for  $h > 2$ . Let  $2m-4 = hq$ , which implies that  $m-1 = -(m-3) + hq$  or  $m-1 \equiv -(m-3) \pmod{h}$ . In this case the equation 3.1 becomes  $(m-3)(x-y) \equiv 0 \pmod{h}$ . Since  $\gcd(m-3, 2m-4) = 1$  or  $2$ , we have  $\gcd(h, m-3) = 1$ , as a result  $x \equiv y \pmod{h}$ , which does not provide a valid labeling.

Step 3. If  $h$  is an odd divisor of either  $m-1$  or  $m-3$ , then  $ST(m)@C_3$  is not  $h$ -magic. Here we observe that since  $\gcd(m-3, m-1) = 1$  or  $2$  and  $h$  is odd, then  $h$  cannot divide both

$m - 3$  and  $m - 1$ . Without loss of generality, we may assume that  $h$  is a divisor of  $m - 3$  and  $\gcd(h, m - 1) = 1$ . In this case, the equation 3.1 becomes  $y \equiv 0 \pmod{h}$ , which does not provide a valid solution. □

**Examples 3.11.**

- (a)  $IM(ST(4)@C_3) = \mathbb{N} - \{2, 3, 4\}$ . Here,  $m = 4$ , and we need to exclude all the divisors  $d > 1$  of  $2m - 4 = 4$ ,  $m - 1 = 3$ , and  $m - 3 = 1$ .
- (b)  $IM(ST(6)@C_3) = \mathbb{N} - \{2, 3, 4, 5, 8\}$ . Here,  $m = 6$ . We need to exclude all the divisors of  $2m - 4 = 8$ ,  $m - 1 = 5$ , and  $m - 3 = 3$ .
- (c)  $IM(ST(14)@C_3) = \mathbb{N} - \{2, 3, 4, 6, 8, 11, 12, 13, 24\}$ . Here,  $m = 14$ . We need to exclude all the divisors of  $2m - 4 = 24$ ,  $m - 1 = 13$ , and  $m - 3 = 11$ .
- (d) The integer-magic spectrum of  $ST(38)@C_{95}$  is

$$\mathbb{N} - \{h \in \mathbb{N} : 2 \leq h \leq 9\} \cup \{12, 18, 24, 35, 36, 37, 72\}.$$

**Theorem 3.12.** *Given any  $n \geq 2$ , there is a graph  $G$  such that  $G$  is not  $h$ -magic for every  $h = 2, 3, \dots, n$ .*

*Proof.* As we observed in the 3.11, the graph  $ST(4)@C_3$  is not  $h$ -magic for  $h = 2, 3, 4$ . So we may assume that  $n \geq 4$ . Let  $\mu$  be the least common multiple of the numbers  $2, 3, \dots, n$ , and consider the graph  $G = ST(m)@C_3$ , where  $m = \frac{\mu + 4}{2}$ . Note that here  $\mu$  is divisible by 4, and so is  $\mu + 4$ , which implies that  $m$  is even. Also, any  $h = 2, 3, \dots, n$ , is a divisor of  $\mu = 2m - 4$ . Therefore, the graph  $ST(m)@C_3$  is not  $h$ -magic. □

We note that the number  $m$ , presented in the proof of the theorem 3.12, might not be the smallest possible answer. For example, in 3.11, we realized that  $ST(38)@C_3$  is not  $h$ -magic for all  $2 \leq h \leq 9$ , while it is 10-magic. In this case, number 38 works, while the least common multiple of the numbers  $2, 3, \dots, 9$  is  $\mu = 2520$ , and the number provided by this theorem is  $m = 1262$ .

We conclude this paper by the following problems:

**Problem 3.13.** *For any positive integer  $n \in \mathbb{Z}_+$  find the smallest  $m \in \mathbb{N}$  such that the graph  $ST(m)@C_3$  is not  $h$ -magic for all  $2 \leq h < n$ .*

**Problem 3.14.** *As examined in 3.11, the graph  $ST(6)@C_3$  has the property that it is 6-magic but it is not  $h$ -magic for all  $2 \leq h \leq 5$ . Find all  $m \in \mathbb{N}$  such that  $ST(m)@C_3$  is not  $h$ -magic, for all  $2 \leq h < m$ .*

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