IC-COLORINGS AND IC-INDICES OF GRAPHS

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ABSTRACT. Given a coloring $f : V(G) \to I\!N$ of graph G and any subgraph $H \subset G$ we define $f_s(H) = \sum_{v \in V(H)} f(v)$. In particular, we denote $f_s(G)$ by $S(f)$. The coloring f is called an IC-coloring if for any integer $k \in [1, S(f)]$ there is a connected subgraph $H \subset G$ such that $f_s(H) = k$. Also, we define the IC-index of G to be

$$M(G) = \max\{ S(f) \ : \ f \text{ is an IC-coloring of } G \}.$$

In this paper we examine some well-known classes of graphs and determine their IC-indices. In addition, several conjectures are proposed.

Key Words: IC-coloring, IC-index of a graph.

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1. INTRODUCTION

Suppose a country wishes to issue a block of four stamps, as illustrated in Figure 1, with denominations a, b, c, and d. How should the values a, b, c, and d be assigned so that one could remove a connected group of stamps of total value k for each $k = 1, 2, \cdots, 10$?

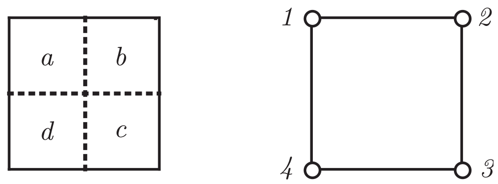

FIGURE 1

This problem is equivalent to assigning positive integer labels to the vertices of C_4 in such a way that for each integer $k = 1, 2, \cdots, 10$ there is a connected induced subgraph whose labels sum to k.

Given a graph G any function $f : V(G) \to I\!N$ is called a *coloring* of G. Let f be a coloring of G and let H be a subgraph of G. We define

$$f_s(H) = \sum_{v \in V(H)} f(v).$$

In particular, when there is no ambiguity about the graph G, we denote $f_s(G)$ by $S(f)$. A function $f : V(G) \to I\!N$ is called an *IC-coloring* of G if for any integer $k \in [1, S(f)]$ there is a connected subgraph H of G such that $f_s(H) = k$. Any connected graph G admits IC-coloring; For example,

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the function $f : V(G) \to I\!N$ defined by $f(v) = 1$ for all $v \in V(G)$ is an IC-coloring with the sum $S(f) = n$. Here, for any integer $k \in [1, n]$, a connected subgraph H of G with exactly k vertices will satisfy the condition $f_s(H) = k$.

The *IC-index* of a graph G, denoted by $M(G)$, is defined to be

$$M(G) = \max\{ S(f) : f \text{ is an IC-coloring of } G \}.$$

Any IC-coloring $\lambda : V(G) \to I\!N$ for which $\lambda_s(G) = M(G)$ will be called a *maximum* IC-coloring of G. A maximum IC-coloring of a graph is not necessarily unique, as illustrated in Figure 2. However, in any IC-coloring of a graph G, one has to use the color 1 and assign this color to a vertex that is not a cut-vertex; otherwise, the number $k = S(f) - 1$ cannot be obtained from any connected subgraph of G.

FIGURE 2. Two distinct maximum IC-colorings of C_4.

The problem of finding IC-colorings of finite graphs is related to the postage stamp problem in number theory, which has been extensively studied in literature [1, 5, 7, 8, 10, 11, 12, 14, 15]. In 1992 Glenn Chappel formulated IC-colorings as "subgraph sums problem". He observed the IC-index of cycle C_n is bounded above by $n(n-1)+1$. Also, Yuzuru Hiraga has reported that $M(C_n)$ has been discussed in a Japanese magazine in 1982, and it is known that $M(C_n) = n^2 - n + 1$ if n is a prime power p^r, (see www.math.uiuc.edu/˜west/pcol/pcol22.ps). This statement is not quite accurate, because we already know that equality does not hold for $n = 7$.

In 1995 Penrice [13], introduced the concept of stamp covering of G as follows: for an integer $k > 0$, a labeling $f : V(G) \to I\!N$ is called a k-labeling if for every integer j, $1 \le j \le k$, there exists a connected induced subgraph H of G with $f_s(H) = j$. Then $M(G)$, the stamp covering number of G, is the largest $k \in I\!N$ such that G has a k-labeling. Furthermore, he showed that

(1) $M(K_n) = 2^n - 1$.
(2) $M(K_3 - e) = 6$, and for every $n \ge 4$, $M(K_n - e) = 2^n - 3$.
(3) $M(K_{1,n}) = 2^n + 2$, for all $n \ge 2$.
(4) If $n \in \{3, 4, 5, 6, 8, 9\}$, then $M(C_n) = n(n-1)+1$. Equality does not hold for $n = 7$.
(5) For positive integer $n \ge 4$, $(n^2 + 6n - 4)/4 \le M(P_n) \le \binom{n+1}{2} - 1$.

In 1998 John Fink [4], considered the following labeling problem:

> Given a connected graph G having p vertices, is it possible to assign labels $1, 2, \cdots, p$ to the vertices in such a way that for each $k = 1, 2, \cdots, p(p+1)/2$, G contains a connected subgraph whose labels sum to k?

Fink called graphs that admit such a labeling sum-saturable graphs. He showed that all cycles, Hamiltonian graphs, and complete bipartite graphs are sum-saturable. In addition, Fink proved the following results:

Theorem 1.1. *Let G be a 2-connected graph having either a one-factor (a 1-regular spanning subgraph) or a near one-factor subgraph (a spanning subgraph F with the property that F has a single vertex v of degree 0 and $F - v$ is 1-regular). Then G is sum-saturable.*

Theorem 1.2. *A connected graph G of order p is sum-saturable if it has a vertex v for which*

(i) $\deg(v) \geq 1 + \lceil \log_2(p-1) \rceil$, *and*

(ii) *there is a proper subset S of $N(v)$ such that $|S| = \lceil \log_2(p-1) \rceil$ and $G - S$ is connected.*

In this paper we will examine some well-known classes of graphs and determine their IC-indices. All the graphs will be finite, simple, and undirected. For undefined terms and concepts the reader is referred to [3].

Observation 1.3. *If H is a subgraph of G, then $M(H) \leq M(G)$.*

Observation 1.4. *If $\nu(G)$ is the number of connected induced subgraphs of G, then $M(G) \leq \nu(G)$.*

Proof. Let $H_1, H_2, \cdots, H_\nu$ be the connected subgraphs of G and let $f : V(G) \to I\!N$ be any maximum IC-coloring of G. Since each one of these subgraphs can only generate one positive integer, namely $f_s(H_i) \in [1, M(G)]$, there will be at most $\nu(G)$ distinct consecutive integers starting from 1. The largest possible number, associated to G, would be $\nu(G)$. Therefore, $M(G) \leq \nu(G)$. □

Observation 1.5. *If $M(G) = \mu$ and $w \notin V(G)$, then $M(G + w) \geq 2\mu + 1$.*

Proof. Let $f : V(G) \to I\!N$ be any maximum IC-coloring of G, and define $\phi : V(G) \cup \{w\} \to I\!N$ by

$$\phi(x) = \begin{cases} f(x) & \text{if } x \in V(G) \\ \mu + 1 & \text{if } x = w. \end{cases} \tag{1.1}$$

We claim that ϕ is an IC-coloring of $G+w$ with $S(\phi) = 2\mu+1$. Clearly, $S(\phi) = \phi(w) + \sum_{v \in V(G)} \phi(v) = (\mu+1) + \mu = 2\mu+1$. Now let $k \in [1, 2\mu+1]$ be any positive integer. We will consider the following three cases:

(a) If $k \leq \mu$, then since $f : V(G) \to I\!N$ is an IC-coloring, there is a connected subgraph H of G such that $f_s(H) = k$. Note that H is also a connected subgraph of $G + w$ with $\phi_s(H) = f_s(H) = k$.

(b) If $k = \mu + 1$, then $\phi(\{w\}) = \mu + 1$.

(c) Suppose $\mu + 2 \le k \le 2\mu + 1$. Then we choose the connected subgraph H of G for which $f_s(H) = k - \mu - 1$ and note that $H + w$ is a connected subgraph of $G + w$ with $\phi(H + w) = \phi(w) + f_s(H) = k$.

This shows that $\phi : V(G) \cup \{w\} \to I\!N$ is an IC-coloring and therefore $M(G+w) \ge S(\phi) = 2\mu+1$. $\square$

The inequality in 1.5 is sharp. As we will see in the next section, for complete graphs we have $M(K_{n+1}) = 2M(K_n) + 1$. However, there are cases, when equality does not hold. For example, consider the graph G, in Figure 3, with $M(G) = 12$ (this graph has exactly 12 non-empty connected induced subgraphs). Now, by Observation 1.5, the graph $M(G + w) \ge 25$. Although $G + w$ has an IC-coloring with sum 25, but it is not maximal. In Figure 3, another IC-coloring of $G + w$ with sum 27 is presented.

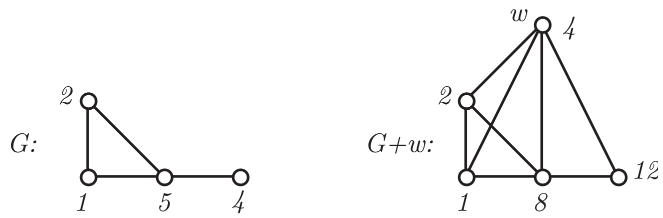

FIGURE 3. An example of G and $G + w$ with $M(G + w) \ne 2M(G) + 1$.

2. Maximum IC-indices of complete graphs and stars

Theorem 2.1. *For any complete graph K_n, we have $M(K_n) = 2^n - 1$.*

Proof. Let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ and define $f : V(K_n) \to I\!N$ by $f(v_i) = 2^{i-1}$. Then $f_s(K_n) = \sum_{i=1}^{n} 2^{i-1} = 2^n - 1$. For any integer $k \in [1, 2^n - 1]$ let $k = \sum_{i=1}^{n} c_i 2^{i-1}$ be its unique binary expansion, and let H be the subgraph of K_n induced by the vertices $\{v_i \ : \ c_i = 1\ \}$. Then $f_s(H) = k$, which shows f is an IC-coloring of K_n, and $M(K_n) \ge 2^n - 1$.

On the other hand, since K_n has exactly $2^n - 1$ non-empty connected induced subgraphs, by 1.4, we have $M(K_n) \le 2^n - 1$. Therefore, $M(K_n) = 2^n - 1$. $\square$

For any $n \ge 1$, the complete bipartite graph $K(1, n)$ is called *star* and is denoted by $ST(n)$. Note that $K(1, 1)$ is the same as P_2, the path of order two, with $M(P_2) = 3$. Similarly, $K(1, 2)$ is the same as P_3, the path of order three, and $M(P_3) = 6$. Also, $M(ST(3)) = 10$. The maximal IC-coloring of $ST(2)$ and $ST(3)$ are illustrated in Figure 4.

Theorem 2.2. *For any $n \ge 2$, $\quad M(ST(n)) = 2^n + 2$.*

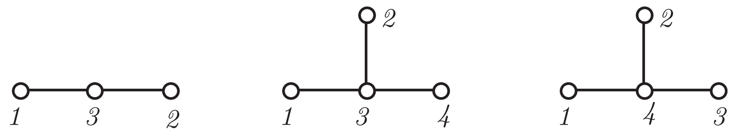

FIGURE 4. Maximal IC-colorings of $ST(2)$ and $ST(3)$.

Proof. Clearly, the statement of this theorem is true for $n = 2, 3$. So let $n \geq 4$, let $c = v_0$ be the central vertex, and let $v_1, v_2, \cdots, v_n$ be the end-vertices of $ST(n)$.

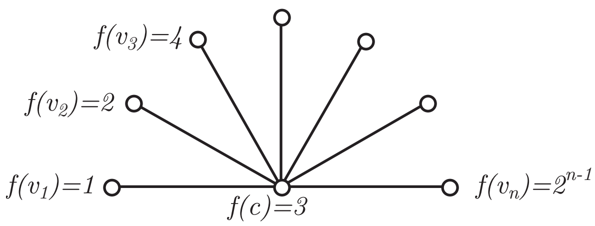

FIGURE 5. A typical maximal IC-coloring of $ST(n)$.

We define a coloring $f : V(ST(n)) \to I\!N$ by $f(c) = 3,\ f(v_i) = 2^{i-1}$. We claim that f is an IC-coloring of $ST(n)$, with $S(f) = 3 + \sum_{i=1}^{n} 2^{i-1} = 2 + 2^n$. For any integer $k \in [1, 2^n + 2]$, if $k = 1, 2$ or 3, then we simply choose the vertex that is labeled k. Now let $k \geq 4$ and let $k - 3 = \sum_{i=0}^{n} c_i 2^i$ be the binary expansion of $k - 3$, where $c_i = 0, 1$. Consider the subgraph H of $ST(n)$ induced by the vertices $\{v_0\} \cup \{v_i : c_i = 1\}$. Then H is connected with $f_s(H) = k$. This shows that f is an IC-coloring and therefore $M(ST(n)) \geq 2^n + 2$.

Now, it is enough to show that $M(ST(n)) \leq 2^n + 2$. Consider a maximum IC-coloring $f : V(ST(n)) \to I\!N$ with $\mu = S(f)$. Without loss of generality, we may assume that $f(v_1) \leq f(v_2) \leq \cdots \leq f(v_n)$. First we need the following two claims:

Claim 1. In any maximum IC-coloring of $ST(n)$, the colors of the end-vertices are distinct.

Proof of Claim 1. Assume to the contrary that $f(v_i) = f(v_{i+1}) = \lambda$. Then it is easy to see that the function $\phi : V(ST(n)) \to I\!N$ defined by

$$\phi(x) = \begin{cases} 2\lambda & \text{if } x = v_{i+1}, \\ f(x) & \text{otherwise ;} \end{cases} \tag{2.1}$$

is an IC-coloring of $ST(n)$ with $S(\phi) = S(f) + \lambda$, which contradicts the maximality of f. Therefore, all the colors assigned to the end-vertices of $ST(n)$ must be distinct.

Claim 2. $f(c) \leq 3$.

Proof of Claim 2. We proceed by contradiction. First let $f(c) = a \geq 5$. Then to produce any one of the numbers $k = 1, 2, \cdots, a - 1$ by a connected subgraph of $ST(n)$, we let $f(v_i) = i \quad (i =$

$1, 2, \cdots, a-1$). As a result, the total contribution of the vertices $c, v_1, v_2, \cdots, v_{a-1}$ to the sum $\mu = S(f)$ will be $a(a+1)/2$ and the colors of these vertices will generate the consecutive positive integers from 1 through $a(a+1)/2$. Now we define the function $\psi : V(ST(n)) \to I\!N$ by

$$\psi(x) = \begin{cases} 3 & \text{if } x = c, \\ 2^{i-1} & \text{if } x = v_i \text{ and } i < a, \\ f(x) & \text{otherwise .} \end{cases} \tag{2.2}$$

Then ψ is a coloring of $ST(n)$ with $S(\psi) = 2^{a-1} + 2 + \sum_{i=a+1}^{n} f(v_i) > \mu = S(f)$. To show that ψ is an IC-coloring, let k be any positive integer in $[1, S(\psi)]$, and consider the following three cases:

Case I. $k \le 2^{a-1} + 2$. Then, as we saw in the first part that k can be produced by a connected induced subgraph of $ST(n)$ with only the vertices $c, v_1, \cdots, v_{a-1}$ involved.

Case II. $2^{a-1} + 2 < k \le S(f)$. Then, in the f-coloring of $ST(n)$, k has a representation as sum of colors of certain vertices.

(a) If in this representation, none of the vertices $v_1, v_2, \cdots, v_{a-1}$ is involved, then the same representation will also work in ψ-coloring of $ST(n)$.

(b) Suppose in this representation, the vertices $v_{i_1}, v_{i_2}, \cdots, v_{i_p}$ are involved, where $i_1 < \cdots < i_p \le a-1$. Then $k = m + f(c) + \sum_{r=1}^{p} f(v_{i_r})$, where m is the contribution from other colors. Again, as we saw earlier that the number $f(c) + \sum_{r=1}^{p} f(v_{i_r})$ can be generated by the ψ-coloring of the vertices $c, v_1, \cdots, v_{a-1}$. If we replace $f(c) + \sum_{r=1}^{p} f(v_{i_r})$ by the ψ-coloring of the vertices $c, v_1, \cdots, v_{a-1}$, we will get the ψ representation of k by a connected subgraph of $ST(n)$.

Case III. $S(f) < k \le S(\psi)$. Then $k - \sum_{i=a}^{n} f(v_i)$ is a positive integer in $[1, 2^{a-1} + 2]$ and has ψ-representation using only the ψ-coloring of the vertices $c, v_1, \cdots, v_{a-1}$, say $k - \sum_{i=a}^{n} f(v_i) = \sum_{r=1}^{p} \psi(v_{i_r})$, where $1 \le i_r \le a-1$. Therefore, $k = \sum_{r=1}^{p} \psi(v_{i_r}) + \sum_{i=a}^{n} f(v_i)$, and k can be written as sum of ψ-colors of a connected induced subgraph of $ST(n)$.

This shows that $f(c) \le 4$. For the case of $n = 3$, as illustrated in Figure 4, the color of c can be 3 or 4. But in both cases the index will be the same and we can only consider the case of $f(c) = 3$. When $n > 3$, one can easily show that the choice of $f(c) = 4$ will generate a smaller sum. Therefore, the colors of the end-vertices are distinct and $f(c) \le 3$. We also observe that $c = v_0$, the central vertex of $ST(n)$, is a cut vertex and in a maximum IC-coloring of stars, in order to generate the integers $\mu - 1$ and $\mu - 2$ one has to label two end-vertices by numbers 1 and 2, as illustrated in Figure 5, which will also take care of $\mu - 3$. Similarly, to generate $\mu - 4$ by a connected induced subgraph of $ST(n)$, one needs to have $f(v_3) = 4$, which will also take care of $\mu - 5$, $\mu - 6$, and $\mu - 7$. Using the numbers $\mu - k$ and induction, one can easily show that $f(v_i) = 2^{i-1}$. Finally, since $f(v_i) = 2^{i-1}$, by choosing $f(c) = 1, 2$, or 3, we will have three different IC-colorings for $ST(n)$. Clearly, the choice of $f(c) = 3$ will provide the maximum sum with $S(f) = 2^n + 2$. Therefore, $\mu \le 2^n + 2$. □

Corollary 2.3. *If $\Delta = \Delta(G)$ is the maximum degree of a connected graph G, then $M(G) \geq 2^{\Delta}+2$.*

If we choose n copies of P_2 and identify one of the end vertices of these n copies of P_2, the resulting graph is the same as star $ST(n)$. From this point of view, one can define $ST(n;\, b_1, b_2, \cdots, b_n)$ to be the graph obtained by identifying one of the end vertices of n paths $P_{b_1}, \cdots, P_{b_n}$. We use the notation $ST(n;\, b^n)$ to denote the graph obtained by identifying one of the end vertices of n copies of P_b, the path of order b.

Theorem 2.4. *For $n \geq 3$, the IC-index of $ST(n;\, 3^n)$ is at least $3^n + 3$.*

Proof. One IC-coloring of $ST(n;\, 3^n)$ with the sum $3^n + 3$ is illustrated in the Figure 6. □

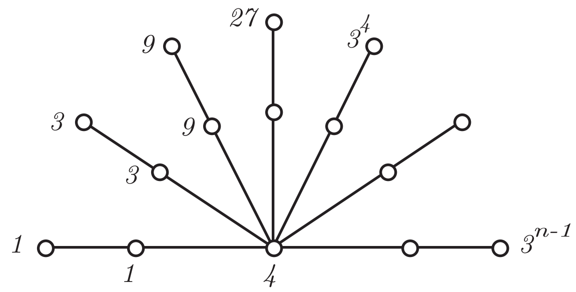

FIGURE 6. An IC-coloring of $ST(n;\, 3^n)$ with sum $3^n + 3$.

Conjecture 2.5. *For $n \geq 4$, the IC-index of $ST(n;\, 3^n)$ is $3^n + 3$.*

Problem 2.6. *Show that the IC-index of $ST(n;\, b^n)$ is at least $b^n + b$.*

Problem 2.7. *Find the IC-index of the graph $ST(n;\, b_1, b_2, \cdots, b_n)$.*

Theorem 2.8. *For any $n \geq 2$, the IC-index of the complete bipartite graph $K(2, n)$ is $3 \cdot 2^n + 1$.*

Proof. Clearly, the statement is true for $n = 2$. So we will assume that $n \geq 3$. Let $P = \{x_1, x_2\}$ and $Q = \{v_1, v_2, \cdots, v_n\}$ be the partite sets of $G = K(2, n)$. The function $f : V(G) \rightarrow I\!N$ defined by $f(x_1) = 2,\ f(x_2) = 4,\ f(v_1) = 1$, and $f(v_j) = 3 \cdot 2^{j-1}$ $(2 \leq j \leq n)$ is an IC-coloring of G with sum $3 \cdot 2^n + 1$.

To show that f is an IC-coloring of G, consider any integer $k \in [3, 3 \cdot 2^n + 1]$ and let $k - 2 = 3q + r$, where $0 \leq q < 2^n$ and $r = 0, 1, 2$. Also, let $q = \sum_{j=0}^{n-1} c_j 2^j$ be the binary expansion of q and $A = \{v_{j+1} \;:\; c_j = 1\}$.

If $r = 0$, then the connected subgraph H_0 of G induced by the vertices $A \cup \{x_1\}$ has the property that $f_s(H_0) = 3q + 2 = k$.

If $r = 2$, then the connected subgraph H_2 of G induced by the vertices $A \cup \{x_2\}$ has the property that $f_s(H_2) = 3q + 4 = k$.

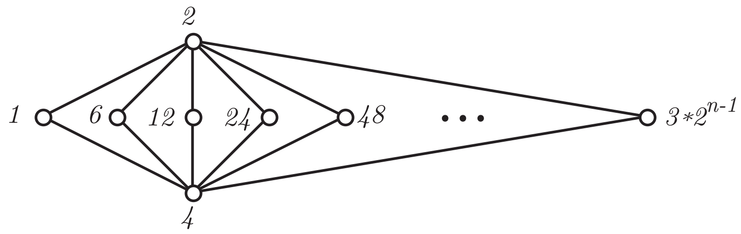

FIGURE 7. An IC-coloring of $K(2,n)$ with sum $3 \cdot 2^n + 1$.

If $r = 1$, then depending on whether v_1 is a member of A or not, the connected subgraph $H_1 =< (A - \{v_1\}) \cup \{x_1, x_2\} >$ or $K_1 =< A \cup \{x_1, v_1\} >$ has the property that $f_s(H_1) = f_s(K_1) = 3q + 3 = k$.

Another IC-coloring of G with the same sum is the function $g : V(G) \to \mathbb{N}$ defined by $g(x_1) = 1,\ g(x_2) = 2,\ f(v_j) = 3 \cdot 2^{j-1}\ \ 1 \leq j < n$ and $g(v_n) = 3 \cdot 2^{n-1} + 1$.

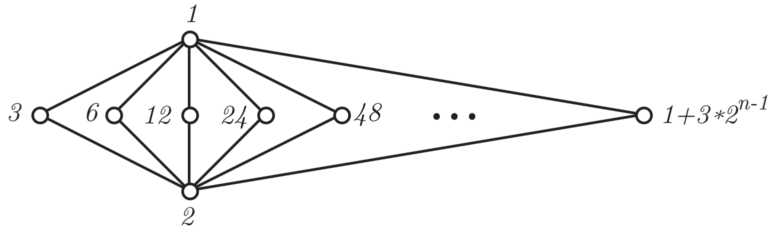

FIGURE 8. Another IC-coloring of $K(2,n)$ with sum $3 \cdot 2^n + 1$.

This shows that $M(G) \geq 3 \cdot 2^n + 1$. We will proceed by contradiction to prove that $M(G) = 3 \cdot 2^n + 1$. Suppose $M(G) > 3 \cdot 2^n + 1$ and let $f : V(G) \to \mathbb{N}$ be a maximum IC-coloring of G. Since the number of connected induced subgraphs of G is $3 \cdot 2^n + n - 1$, we will have

$$3 \cdot 2^n + 1 < S(f) \leq 3 \cdot 2^n + n - 1. \tag{2.3}$$

An immediate consequence of (2.3) is that there are at most $n - 3$ pairs H and H' of distinct connected subgraphs of G with $f_s(H) = f_s(H')$. Now we have the following observation:

Fact 1. Let $T,\ T_1,\ T_2$ be subsets of Q with $|T| \leq 3,\ |T_1| + |T_2| \leq 3$, and $T_1 \cap T_2 = \emptyset$. Then each one of the following five conditions insures that there are at least $n - 2$ pairs of distinct connected subgraphs H and H' with $f_s(H) = f_s(H')$:

(a) $f(x_i) = \sum_{v \in T} f(v)$. Here, $T \neq \emptyset$;

(b) $f(x_i) = f(x_j) + \sum_{v \in T} f(v)$;

(c) $\sum_{v \in T_1} f(v) = \sum_{v \in T_2} f(v)$. Here, $T_1, T_2 \neq \emptyset$;

(d) $f(x_i) + \sum_{v \in T_1} f(v) = \sum_{v \in T_2} f(v)$. Here, $T_2 \neq \emptyset$;

(e) $f(x_i) + \sum_{v \in T_1} f(v) = f(x_j) + \sum_{v \in T_2} f(v)$.

Proof of Fact 1. Since the proofs of (a)-(e) are very similar, we will only present the proof for part (a): For any subset $S \subset Q - T$, the pair of subgraphs induced by the vertices $P \cup S$ and $\{x_j\} \cup S \cup T$ are connected and have the same vertex sum. This will produce at least $2^{n-3} \geq n - 2$ pairs of distinct connected subgraphs H and H' with $f_s(H) = f_s(H')$.

An immediate consequence of Fact 1 is that the numbers $f(x_1), f(x_2), f(v_1), \cdots, f(v_n)$ are all distinct. As a result, without loss of generality, we may assume that $f(x_1) < f(x_2)$ and $f(v_1) < \cdots < f(v_n)$. Also, since $S(f) > 3 \cdot 2^n + 1$, then none of the conditions (a)-(e) of the Fact 1 will be allowed. Now we will consider the following four cases:

Case 1. If $f(v_1) = 1$ and $f(v_2) = 2$, then by Fact 1(a), $f(x_i) \neq 3$. Since we need to cover 3, then $f(v_3) = 3$, which by the Fact 1(c) is not allowed.

Case 2. If $f(x_1) = 1$ and $f(v_1) = 2$, then by Fact 1(b), (d) we have $f(x_2)$, $f(v_2) > 3$.
If $f(x_2) = 4$, then $f(v_2) = 5$, which is not allowed by Fact 1(e).
If $f(v_2) = 4$, then $f(x_2) \neq 5, 6, 7$. Therefore, $f(v_3) = 6$ and this is not allowed by Fact 1(c).

Case 3. Assume $f(x_1) = 2$ and $f(v_1) = 1$. Then by Fact 1, we know that $f(x_2), f(v_2) > 3$. If $f(x_2) > 4$, then we must have $f(v_2) = 4$. Since $f(x_2) \notin \{5, 6, 7\}$, we are forced to have $f(v_3) = 5$, which gives us $f(v_3) = f(v_1) + f(v_2)$ and this is not allowed by Fact 1(c). Therefore, $f(x_2) = 4$ and consequently $f(v_2) = 6$. Now we use strong induction to show that $f(v_k) = 3 \cdot 2^{k-1}$ for all k with $2 \leq k < n$. Clearly this is true for $k = 2$. Assume that $f(v_j) = 3 \cdot 2^{j-1}$ for all j with $2 \leq j \leq k < n - 1$. We wish to show that $f(v_{k+1}) = 3 \cdot 2^k$. Let $M = f(x_1) + f(x_2) + \sum_{j=1}^{k} f(v_j) = 3 \cdot 2^k + 1$. If $f(v_{k+1}) \geq M$, then number $1 + f(v_{k+1})$ cannot be produced by any connected subgraph of G induced by the vertices $x_1, x_2, v_1, \cdots, v_{k+1}$, and one has to assign this number to vertex v_{k+2}; that is, $f(v_{k+2}) = 1 + f(v_{k+1}) = f(v_1) + f(v_{k+1})$, which is not allowed by Fact 1(c).

Suppose $f(v_{k+1}) < M - 1$ and define the coloring $\phi : V(G) \to I\!N$ by

$$\phi(x) = \begin{cases} M - 1 & \text{if } x = v_{k+1}, \\ f(x) & \text{otherwise.} \end{cases} \tag{2.4}$$

Then $S(\phi) = S(f) - f(v_{k+1}) + M - 1 > S(f)$. To show that ϕ is an IC-coloring, let p be any integer in $[1, S(\phi)]$.

Subcase I. $1 \leq p \leq S(f)$.

Since f is an IC-coloring of G, there is a connected subgraph H of G such that $f_s(H) = p$. If $v_{k+1} \notin V(H)$, then $\phi_s(H) = p$ and we are done. Suppose $v_{k+1} \in V(H)$, and let $H_1 = V(H) \cap \{x_1, x_2, v_1, \cdots, v_k\}$, and $H_2 = V(H) \cap \{v_{k+2}, \cdots, v_n\}$. We have

$p - f_s(H_2) = f_s(H_1) + f(v_{k+1})$. Since $0 \leq f_s(H_1) \leq M$ and $1 < f(v_{k+1}) < M-1$, we get $1 < p - f_s(H_2) < 2M - 1$. If $1 < p - f_s(H_2) \leq M$, then $p - f_s(H_2) = f_s(H^{'})$, where $H^{'}$ is a connected subgraph of $< H_1 >$. It follows that $p = f_s(H^{'} \cup H_2) = \phi_s(H^{'} \cup H_2)$. If $M < p - f_s(H_2) < 2M - 1$, then $1 < p - f_s(H_2) - (M-1) < M$. Thus $p - f_s(H_2) - (M-1) = f_s(H^{''})$, where $H^{''}$ is a connected subgraph of $< H_1 >$. It follows that $p = f_s(H^{''} \cup H_2) + M - 1 = \phi_s(H^{''} \cup \{v_{k+1}\} \cup H_2)$. Notice that if $V(H^{'}) = \{v_j\}$ (or $V(H^{''}) = \{v_j\}$) for some $j \geq 2$, then we may replace $H^{'}$ (or $H^{''}$) with the graph induced by the vertices $\{x_1, x_2, v_1, \cdots, v_{j-1}\}$. Thus the graphs $H^{'} \cup H_2$ and $H^{''} \cup H_2 \cup \{v_{k+1}\}$ are connected.

Subcase II. $S(f) < p \leq S(\phi)$.

If $p = S(\phi)$, obviously $p = \phi_s(G)$. So, let $p < S(\phi)$ and therefore $0 < S(f) - f(v_{k+1}) + (M-1) - p < M - 1$. It follows that $S(f) - f(v_{k+1}) + (M-1) - p = f_s(G^{'})$, where $G^{'}$ is a connected subgraph of $< H_1 >$. If $\{x_1, x_2\} \subset V(G^{'})$, let $t \geq 1$ be the smallest integer such that $v_t \notin V(G^{'})$. Since $f_s(G^{'}) < M - 1$, such an integer t exists. Then we replace $G^{'}$ with the graph $G^{''}$ induced by $V(G^{'}) \cup \{v_t\} - \{x_1, x_2, v_1, \cdots, v_{t-1}\}$. We have $f_s(G^{''}) = f_s(G^{'})$ and $G - G^{''}$ is a connected graph. If $\{x_1, x_2\}$ is not a subset of $V(G^{'})$, then we know that $G - G^{''}$ is a connected graph, and we have $p = S(\phi) - \phi_s(H^{'}) = f_s(G - H^{'}) = f_s(G - G^{''})$ and we are done.

This proves that ϕ is an IC-coloring of G with $S(\phi) > S(f)$, which is a contradiction to the choice of f. Hence $f(v_{k+1}) = M - 1 = 3 \cdot 2^k$ is our only choice. But, then $S(f) = 3 \cdot 2^n + 1$, which is again a contradiction to the choice of f.

Case 4. Assume $f(x_1) = 1$ and $f(x_2) = 2$. In order to cover 3, we have to have $f(v_1) = 3$. Let k be the smallest number such that $f(v_k) \neq 3 \cdot 2^{k-1}$. Clearly, $2 \leq k \leq n$. If $k = n$, then $f(v_j) = 3 \cdot 2^{j-1}$ for $j = 1, 2, \cdots, n-1$ and $f(v_n) = 3 \cdot 2^{n-1} + 1$. This will provide an IC-coloring of G with $S(f) = 3 \cdot 2^n + 1$. Suppose $k < n$, and let H be the subgraph induced by the vertices $x_1, x_2, v_1, \cdots, v_{k-1}$. Since $f(v_j) = 3 \cdot 2^{j-1}$ for $j = 1, 2, \cdots, k-1$, then $f_s(H) = 3 \cdot 2^{k-1}$ and the numbers 1 through $3 \cdot 2^{k-1}$ are produced by the connected subgraphs of H, which are connected subgraphs of $K(2, n)$ as well. Also, $f(v_k) > f(v_{k-1})$. The number $f(v_k) + f(v_{k-1})$ cannot be produced by the subgraph $K = < x_1, x_2, v_1, \cdots, v_{k-1}, v_k >$, and one has to color v_{k+1} by $f(v_k) + f(v_{k-1})$; that is, $f(v_{k+1}) = f(v_k) + f(v_{k-1})$, which is not allowed by Fact 2(c).

This proves that it is not possible to have an IC-coloring of G with $S(f) > 3 \cdot 2^n + 1$. Therefore, $M(G) \leq 3 \cdot 2^n + 1$ and therefore $M(G) = 3 \cdot 2^n + 1$. □

Although one can use Theorem 2.8 to obtain the IC-index of $K(2, 3)$, we will present an independent proof for this particular case.

Example 2.9. *The IC-index of* $G = K(2,3)$ *is 25.*

Proof. First we observe that $M(G) \geq 25$; because, as illustrated in Figure 9, there is an IC-coloring of G with sum 25. Therefore, it is enough to show that $M(G) \leq 25$.

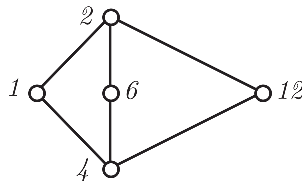

FIGURE 9. An IC-coloring of $G = K(2,3)$ with sum 25.

Next, we observe that G has a total of 26 connected subgraphs, so

$$M(G) \leq 26. \tag{2.5}$$

Now let $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and let $f : V(G) \to I\!N$ be a maximum IC-coloring of G. If $f(v_i) = f(v_j)$ for some $i \neq j$, then $f(G) \leq 25$. Hence, without loss of generality, by renumbering the vertices if needed, we may assume that $1 = f(v_1) < f(v_2) = 2 < \cdots < f(v_5)$. Then $f(v_5) \leq 13$. Because, if $f(v_5) \geq 14$, then the subgraph $H =< v_1, v_2, v_3, v_4 >$ induced by the vertices v_1, v_2, v_3, and v_4 would have to produce the consecutive numbers $1, 2, 3, \cdots, 13$. This means $S(f) = f_s(H) + f(v_5) \geq 13 + 14 = 27$, which contradicts (2.5). Therefore, $f(v_5) \leq 13$.
Now we consider the following two cases.

Case I. The two vertices v_1 and v_2 are not adjacent. Then $f(v_3) = 3$. Here, we observe that $f(v_4) \leq 7$; otherwise, the number 7 cannot be produced. This implies that $f_s(H) \leq 13$. If $f_s(H) \leq 12$, then $f(G) \leq 25$. If $f_s(H) = 13$, then given the fact that $f(v_5) \leq 13$, either number 13 will be repeated, or $f(v_5) = 12$. In either case, $f(G) \leq 25$.

Case II. The two vertices v_1 and v_2 are adjacent. Then $f(v_3) = 4$. Here, we observe that $f(v_4) \leq 8$; otherwise, the number 8 cannot be produced. Since $f(v_5) \leq 13$, we will have three choices for the pair $f(v_4)$ and $f(v_5)$, which are $(6, 13)$, $(7, 12)$ or $(8, 11)$. The cases $(7, 12)$ and $(8, 11)$ are not acceptable; because, either 5 or 6 cannot be produced. In the case of $(6, 13)$, we will have $f_s(H) = 13$ and as discussed in the previous case, $S(f) \leq 25$.

This shows that $S(f) \leq 25$, and therefore $M(G) = 25$. □

Problem 2.10. *Find the IC-index of the complete bipartite graph* $K(m, n)$.

3. IC-indices of other graphs

3.1. **Paths.** Since P_n contains $n(n+1)/2$ connected subgraphs, $M(P_n) \leq n(n+1)/2$.

Also, there is an IC-coloring of P_n, as illustrated in Figure 10, with sum

$$(2 + \lfloor n/2 \rfloor)(n - \lfloor n/2 \rfloor) + \lfloor n/2 \rfloor - 1$$

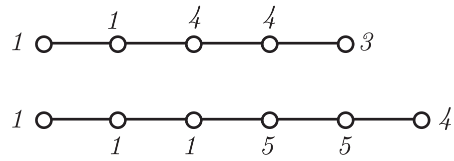

FIGURE 10. IC-coloring of the paths P_5 and P_6.

Theorem 3.1. *For any* $n \in I\!N$, $M(P_n) \geq (2 + \lfloor n/2 \rfloor)(n - \lfloor n/2 \rfloor) + \lfloor n/2 \rfloor - 1$.

3.2. **Double-stars.** Trees of diameter three are called *double-stars.* These graphs have two central vertices plus leaves. We will use $DS(m, n)$ to denote the double-star whose two central vertices have degrees m and n, respectively.

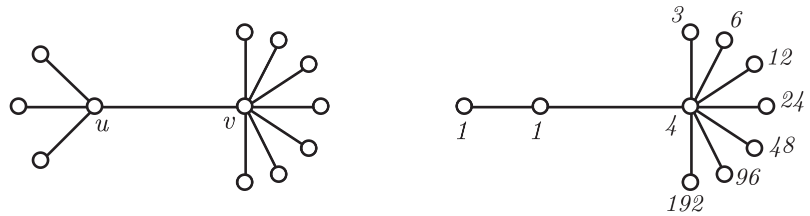

FIGURE 11. Double-star $DS(4, 8)$ and $DS(2, 8)$.

As an example, we know that double-star $DS(2, n)$ has a total of $3 \cdot 2^{n-1} + (n + 2)$ connected induced subgraphs, but an IC-coloring with such sum may not be possible. However, there is an IC-coloring with sum $3(2^{n-1}+1)$, as illustrated in Figure 11. We suspect that this number is indeed its IC-index.

Lemma 3.2. *For* $2 \leq m \leq n$*, the IC-index of* $DS(m, n)$ *is at least* $(2^{m-1} + 1)(2^{n-1} + 1)$.

Proof. Let $G = DS(m, n)$, and let u and v be the central vertices of G with degrees m and n, respectively. Also, let $N(u) = \{u_1, u_2, \cdots, u_m = v\}$ and $N(v) = \{v_1, v_2, \cdots, v_n = u\}$ be the neighbors of these central vertices. Define the function $f : V(G) \to I\!N$ by $f(u) = 1$, $f(u_i) = 2^{i-1}$, $f(v) = 2^{m-1} + 2$, and $f(v_j) = (2^{m-1} + 1) \cdot 2^{j-1}$ for every $1 \leq i \leq m - 1$ and $1 \leq j \leq n - 1$. One can easily show that f is an IC-coloring of G with sum $(2^{m-1} + 1)(2^{n-1} + 1)$. □

Corollary 3.3. *For any triangle-free graph* G *with more than two vertices, we have*

$$M(G) \geq (2^{\Delta-1} + 1)(2^{\delta-1} + 1),$$

where Δ *and* δ *denote the maximum and minimum degrees of* G*, respectively.*

Proof. Let v be a vertex of G with degree Δ, and let u be any vertex adjacent to v. Since the graph is triangle free, the graph induced by u, v, and their neighbors is the double-star $G = DS(d(u), d(v))$. Thus

$$M(G) \geq (2^{d(v)-1} + 1)(2^{d(u)-1} + 1) \geq (2^{\Delta-1} + 1)(2^{\delta-1} + 1).$$

□

Conjecture 3.4. *For $2 \leq m \leq n$, the IC-index of $DS(m, n)$ is $(2^{m-1} + 1)(2^{n-1} + 1)$.*

3.3. **Cycles.** We observe that the graph C_n has $n(n-1)+1$ connected subgraphs. Therefore, by Observation 1.4, $M(C_n) \leq n(n-1)+1$. Also, Fink [4] has presented an IC-coloring of C_n with sum $n(n+1)/2$, which will result to the following theorem:

Theorem 3.5. *For any $n \geq 3$,* $\dfrac{n(n+1)}{2} \leq M(C_n) \leq n(n-1)+1.$

The right hand side of this inequality is sharp; because, as illustrated in Figure 12, we know that $M(C_n) = n(n-1)+1$ for $n = 3, 4, 5, 6$. But we have not come up with a systematic way of finding a maximal IC-coloring of C_n.

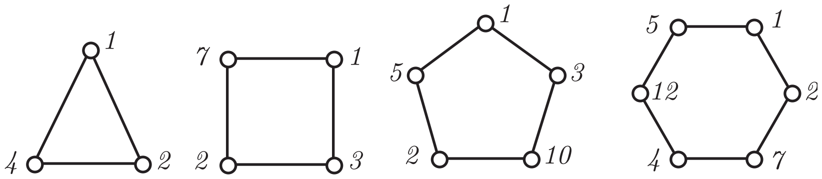

FIGURE 12. Maximal IC-colorings of C_3, C_4, C_5, and C_6.

The lower bound can easily be improved by at least by $\lfloor \frac{n-1}{2} \rfloor$.

3.4. **Wheels.** Wheels with n spokes, denoted by W_n, are obtained by the join operation $C_n + K_1$.

Theorem 3.6. *For any $n \geq 3$, the IC-index of W_n satisfies the following inequalities:*

$$2^n + 2 \leq M(W_n) \leq 2^n + n(n-1) + 1.$$

Proof. Since $\Delta(W_n) = n$, by Corollary 2.3, we have $M(W_n) \geq 2^n + 2$, which proves the lower bound. To prove the upper bound, we need to count the number of connected induced subgraphs of W_n. We noticed earlier that C_n has $n(n-1)+1$ connected subgraphs. Thus W_n has exactly $n(n-1)+1$ connected subgraphs that do not use the vertex K_1. Any subset of $V(C_n)$ together with K_1 forms a connected induced subgraph of W_n. Thus there are exactly 2^n connected induced subgraphs that use the vertex K_1. Therefore, W_n has $2^n + n(n-1)+1$ connected induced subgraphs and by Observation 1.4 we have $M(W_n) \leq 2^n + n(n-1)+1$. □

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