The Friendly Index Set of $P_m \times P_n$

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Abstract

For a graph $G = (V, E)$ and a binary labeling (coloring) $f : V(G) \to \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. f is said to be friendly if $|v_f(1) - v_f(0)| \leq 1$. The labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \to \mathbb{Z}_2$ defined by $f^*(xy) = |f(x) - f(y)|\ \forall xy \in E(G)$. Let $e_f(i) = |{f^*}^{-1}(i)|$. The friendly index set of the graph G, denoted by $FI(G)$, is defined by

$$FI(G) = \{|e_f(1) - e_f(0)| : \ f \text{ is a friendly vertx labeling of } G\ \}.$$

In this paper we determine the friendly index set of $P_m \times P_n$.

Key Words: Labeling; cordial graphs; friendly labeling; friendly-index set.

AMS Subject Classification: 05C78

1 Introduction

In this paper all graphs $G = (V, E)$ are connected, finite, simple, and undirected. For graph theory notations and terminology not described in this paper, we refer the readers to [5]. Let $G = (V, E)$ be a graph and $f : V(G) \to \mathbb{Z}_2$ a vertex labeling (coloring) of G. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. The labeling f is said to be *friendly* if $|v_f(1) - v_f(0)| \leq 1$.

Any vertex labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \to \mathbb{Z}_2$ defined by $f^*(xy) = |f(x) - f(y)|\ \forall xy \in E(G)$. For $i \in \mathbb{Z}_2$, let $e_f(i) = |{f^*}^{-1}(i)|$. The number $N(f) = |e_f(1) - e_f(0)|$ is called the *friendly index* of f. The *friendly index set* of the graph G, denoted by $FI(G)$, is defined by

$$FI(G) = \{N(f) : \ f \text{ is a friendly coloring of } G\}.$$

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A friendly coloring $f : V(G) \to \mathbb{Z}_2$ is called a *maximum friendly coloring* of G if its friendly index is the maximum element of $FI(G)$. Note that if $f : V(G) \to \mathbb{Z}_2$ is a friendly coloring, so is its inverse coloring $g : V(G) \to \mathbb{Z}_2$ defined by $g(v) = 1 - f(v)\ \forall v \in V(G)$. Moreover, $N(g) = N(f)$.

In 1978, Cahit [2, 3, 4] introduced the concept of cordial labeling as weakened version of the less tractable graceful and harmonious labeling. A graph G is said to be cordial if it admits a friendly labeling with index 0 or 1. Hovay [7], later generalized the concept of cordial graphs and introduced A-cordial labelings, where A is an abelian group. A graph G is said to be A-cordial if it admits a labeling $f : V(G) \to A$ such that for every $i, j \in A$,

$$|v_f(i) - v_f(j)| \le 1 \text{ and } |e_f(i) - e_f(j)| \le 1.$$

Cordial graphs have been studied extensively. Interested readers are referred to a number of relevant papers in the literature that are mentioned in the bibliography section, including [1, 6, 8, 9, 11, 14]. In this paper, we focus on the group $A = \mathbb{Z}_2$, and determine the friendly index set of $P_m \times P_n$. As mentioned earlier, the friendly index set of a graph G, denoted by $FI(G)$, is defined by $\{N(f) = |e_f(1) - e_f(0)| : f \text{ is friendly coloring of } G\}$. When there is no ambiguity we drop the subscript f. Note that if 0 or 1 is in $FI(G)$, then G is cordial. Thus the concept of friendly index sets can be viewed as a generalization of cordiality. First we state known results from [10], [12], and [13] to be used in the following section.

Theorem 1.1. [12] *For a graph G with q edges, $FI(G) \subset \{q - 2i : 0 \le i \le \lfloor \frac{q}{2} \rfloor\}$.*

Theorem 1.2. [12] *For a cycle, C_n with $n \ge 3$, $FI(C_{2n}) = \{2n - 4i : 0 \le i \le \lfloor \frac{n}{2} \rfloor\}$ and $FI(C_{2n+1}) = \{2n + 1 - 2i : 1 \le i \le n\}$.*

Lemma 1.3. [12] *For any labeling $f : V(C_n) \to \mathbb{Z}_2$ (not necessarily friendly) $e_f(1)$ is even.*

Theorem 1.4. [10] *The cartesian product of an arbitrary number of paths is cordial.*

Examples 1.5. By Theorem 1.1, $FI(P_2 \times P_6) \subset \{16 - 2i : i = 0, 1, 2, \cdots, 8\}$. Figure 1 shows that $\{16 - 2i : i = 0, 2, 3, \cdots, 7\} \subset FI(P_2 \times P_6)$. In addition, by Theorem 1.4, $0 \in FI(P_2 \times P_6)$. It remains to be shown whether $14 \in FI(P_2 \times P_6)$. We note, however, that for $14 \in FI(P_2 \times P_6)$, there must be a friendly labeling $f : V(P_2 \times P_6) \to \mathbb{Z}_2$ for which either $e_f(1) = 1$ or $e_f(0) = 0$. This, however, contradicts Lemma 1.3, since such an edge labeled 0 or 1 must be contained in some cycle, C_4, on $P_2 \times P_6$. Therefore, $FI(P_2 \times P_6) = \{16 - 2i : i = 0, 2, 3, \cdots, 8\}$.

2 The Friendly Index Set of $P_m \times P_n$

In [12], Lee and Ng show that the friendly index set of $P_2 \times P_n$ with q edges achieves all possible friendly indices as implied by Theorem 1.1, except for $q-2$. In this section, we show that this result

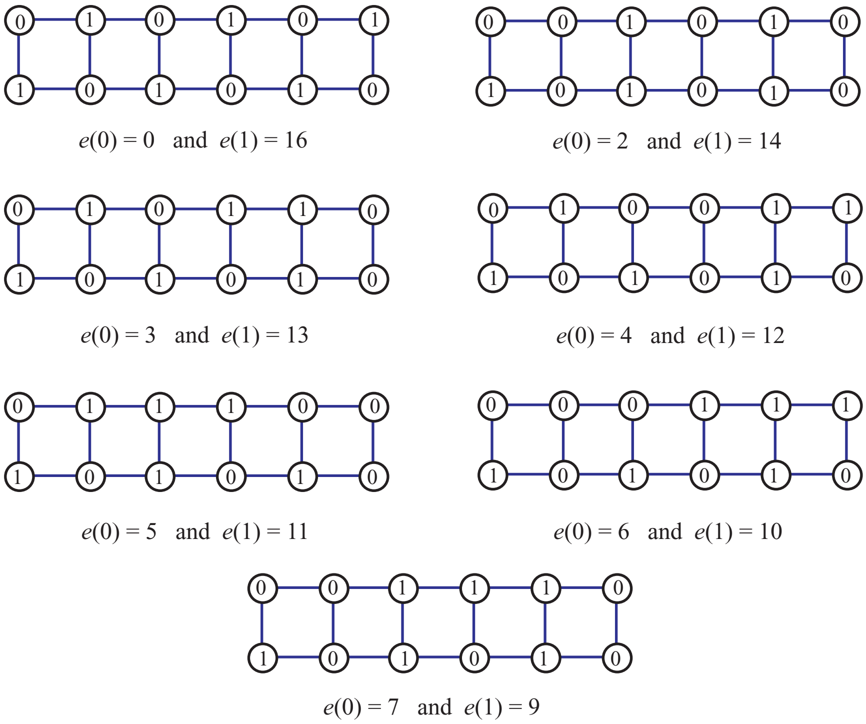

Figure 1: Friendly labelings of $P_2 \times P_6$.

need not extend for the grid $P_m \times P_n$ with $m, n \geq 3$ and evaluate the friendly index set of this graph.

We begin this section by discussing the friendly-index set of $P_2 \times P_n$ and provide an additional condition, which proves to be useful in finding the friendly index set of $P_m \times P_n$. In particular, it serves as the base case for an inductive argument for $P_m \times P_n$. A discussion on the conditions needed for $m(n-1) + n(m-1) - 4$ to be in the friendly index set of $P_m \times P_n$ follows. Finally, we show that the friendly-index set of $P_m \times P_n$ contains certain values by induction. First, consider an illustration of $P_2 \times P_n$ (a ladder) given in Figure 2.

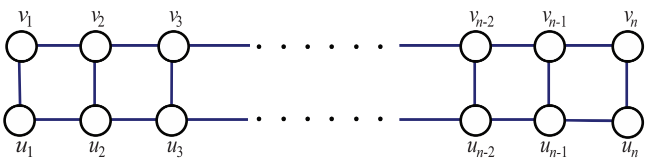

Figure 2: An illustration of $P_2 \times P_n$.

The vertices $v_1, v_2, \cdots, v_n$ are defined to be the top vertices, while the vertices $u_1, u_2, \cdots, u_n$ will be named the bottom vertices of $P_2 \times P_n$, in this paper. A labeling $f : V(P_2 \times P_n) \to \mathbb{Z}_2$ with the property that $f(u_1) = f(u_3) = \cdots = f(u_{2\lceil \frac{n}{2} \rceil - 1}) = 1$ and $f(u_2) = f(u_4) = \cdots = f(u_{2\lceil \frac{n}{2} \rceil}) = 0$ is said to be alternating-bottom labeling. If in addition $f(v_1) = f(v_3) = \cdots = f(v_{2\lceil \frac{n}{2} \rceil - 1}) = 0$ and $f(v_2) = f(v_4) = f(v_{2\lceil \frac{n}{2} \rceil}) = 1$ then we say that the labeling is the maximum-friendly labeling of a ladder and $e_f(1) - e_f(0) = 3n - 2$.

Lemma 2.1. *$P_2 \times P_n$ has $3n-2$ as its maximum friendly index.*

Lemma 2.2. $3n - 4 \notin FI(P_2 \times P_n)$.

Proof. The result follows immediately from Lemma 1.3. □

Lemma 2.3. *$P_2 \times P_n$ is cordial. In particular, for any $n \in \mathbb{N}$, $0 \in FI(P_2 \times_{2n})$ and $1 \in (P_2 \times P_{2n+1})$.*

Lemma 2.4. $3n - 6 \in FI(P_2 \times P_n)$

Proof. Consider the labeling $f : V(P_2 \times P_n) \to \mathbb{Z}_2$ defined by

$$f(v_j) = \begin{cases} 0 & \text{if } j \text{ is even or } j = 1; \\ 1 & \text{if } j \text{ is odd and } j \geq 3; \end{cases}$$

and $f(u_j) = 1 - f(v_j)$. It is easy to see that for this labeling $e_f(0) = 2$. Therefore, $3n - 6 \in FI(P_2 \times P_n)$. □

Theorem 2.5. $FI(P_2 \times P_{2n}) = \{6n - 2 - 2i : i = 0, 2, 3, \cdots, 3n - 1\}$. *Moreover, each element of the friendly index, with the possible exception of 0 and $6n-6$, can be obtained with an alternating-bottom labeling f, such that $e_f(1) \geq e_f(0)$.*

Proof. Let $3 \leq i \leq 3n - 2$. To show $6n - 2 - 2i \in FI(P_2 \times P_{2n})$ it is enough to present a labeling $f : V(P_2 \times P_{2n}) \to \mathbb{Z}_2$ for which $e_f(0) = i$. We consider the following cases:
Case 1. i is odd.
Subcase 1a. $i \not\equiv 1 \pmod 6$.

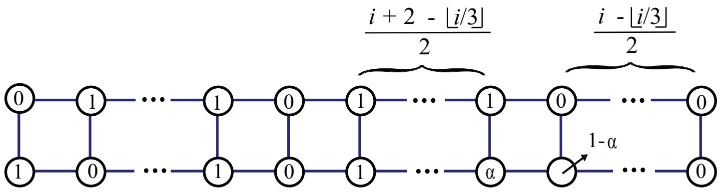

Figure 3: An illustration of labeling described in case 1a.

Define $f : V(P_2 \times P_{2n}) \to \mathbb{Z}_2$, with alternating-bottom labeling, by

$$f(v_j) = \begin{cases} 0 & \text{if } j \in \{1, 3, \cdots, 2n - (i + 1 - \lfloor \frac{i}{3} \rfloor)\}; \\ 1 & \text{if } j \in \{2, 4, \cdots, 2n - (i + 1 - \lfloor \frac{i}{3} \rfloor) - 1\}; \\ 0 & \text{if } j \in \{2n - \frac{i - \lfloor \frac{i}{3} \rfloor}{2} + 1, \cdots, 2n\}; \\ 1 & \text{if } j \in \{2n - (i + 1 - \lfloor \frac{i}{3} \rfloor) + 1, \cdots, 2n - \frac{i - \lfloor \frac{i}{3} \rfloor}{2}\}. \end{cases}$$

A graphical depiction of this labeling is shown in Figure 3, where

$$\alpha = \begin{cases} 1 & \text{if } i \equiv 3 \pmod 6; \\ 0 & \text{otherwise.} \end{cases}$$

Note that there are $\lfloor \frac{i}{3} \rfloor + 1$ edges connecting the two paths that are labeled 0. Additionally, there are

$$\left(\frac{i+1-\lfloor \frac{i}{3} \rfloor + 1}{2} - 1 \right) + \left(\frac{i+1-\lfloor \frac{i}{3} \rfloor - 1}{2} - 1 \right) = i - \lfloor \frac{i}{3} \rfloor - 1$$

top edges labeled 0, showing that $e_f(0) = i$.

Subcase 1b. $i \equiv 1 \pmod 6$.

In this case $i - 2 \not\equiv 1 \pmod 6$, as a result there is a labeling f, as described above, for which $e_f(0) = i - 2$. Now define $g : V(P_2 \times P_{2n}) \to \mathbb{Z}_2$ with alternating-bottom labeling by

$$g(v_j) = \begin{cases} 1 - f(v_j) & \text{if } j \in \{2n - (i - 2 - \lfloor \frac{i-2}{3} \rfloor), 2n - (\frac{i-2-\lfloor \frac{i-2}{3} \rfloor - 2}{2})\}; \\ f(v_j) & \text{otherwise.} \end{cases}$$

For such a labeling, there are $i - 2 - \lfloor \frac{i-2}{3} \rfloor - 1$ top edges labeled 0 and $\lfloor \frac{i-2}{3} \rfloor + 1 + 2$ edges connecting the two paths that are labeled 0, so that $e_g(0) = i$.

Case 2. i is even.

Subcase 2a. Assume that $i \not\equiv 2 \pmod 6$. We denote the top vertices by $v_1, v_2, \cdots, v_n$ and define $f : V(P_2 \times P_{2n}) \to \mathbb{Z}_2$, with alternating-bottom labeling, by

$$f(v_j) = \begin{cases} 0 & \text{if } j \in \{1, 3, \cdots, 2n - (i + 1 - \lfloor \frac{i-1}{3} \rfloor) - 1\}; \\ 1 & \text{if } j \in \{2, 4, \cdots, 2n - (i + 1 - \lfloor \frac{i-1}{3} \rfloor)\}; \\ 0 & \text{if } j \in \{2n - (i + 1 - \lfloor \frac{i-1}{3} \rfloor) + 1, \cdots, 2n - \frac{i+1-\lfloor \frac{i-1}{3} \rfloor}{2}\}; \\ 1 & \text{if } j \in \{2n - \frac{i+1-\lfloor \frac{i-1}{3} \rfloor}{2} + 1, \cdots, 2n\}. \end{cases}$$

A graphical depiction of this labeling is shown in Figure 4, where

$$\alpha = \begin{cases} 1 & \text{if } i \equiv 3 \pmod 6; \\ 0 & \text{otherwise.} \end{cases}$$

Here there are $\lfloor \frac{i-1}{3} \rfloor + 1$ edges connecting the two paths that are labeled 0. In addition, there are

$$\left(\frac{i+1-\lfloor \frac{i-1}{3} \rfloor}{2} - 1 \right) + \left(\frac{i+1-\lfloor \frac{i-1}{3} \rfloor}{2} - 1 \right) = i - \lfloor \frac{i-1}{3} \rfloor - 1$$

top edges labeled 0, showing that $e_f(0) = i$

Subcase 2b. Suppose $i \equiv 2 \pmod 6$.

In this case $i - 2 \not\equiv 2 \pmod 6$. As a result, there is a labeling f, as described in Case 2a, for which $e_f(0) = i - 2$. Now define $g : V(P_2 \times P_{2n}) \to \mathbb{Z}_2$, with alternating-bottom labeling, by

$$g(v_j) = \begin{cases} 1 - f(v_j) & \text{if } j \in \{2n - (i - 2 - \lfloor \frac{i-3}{3} \rfloor), 2n - \frac{i-1-\lfloor \frac{i-3}{3} \rfloor}{2} + 1\}; \\ f(v_j) & \text{otherwise.} \end{cases}$$

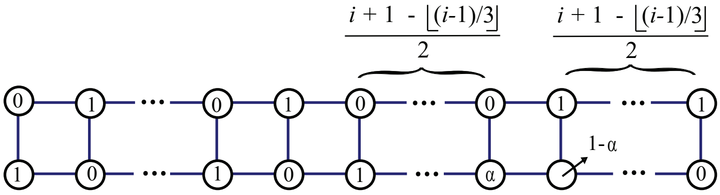

Figure 4: An illustration of the labeling described in case 2a.

For such a labeling, there are $i-2-\lfloor\frac{i-3}{3}\rfloor-1$ top edges labeled 0 and $\lfloor\frac{i-3}{3}\rfloor+1+2$ edges connecting the two paths that are labeled 0, so that $e_g(0)=i$.
Applying Lemmas 2.1, 2.2, 2.3, 2.4, and Theorem 1.1 concludes the proof. □

Using similar labelings, as discussed in the proof of Theorem 2.5, one can show the following result:

Theorem 2.6. *$FI(P_2 \times P_{2n+1}) = \{6n+1-2i : i = 0, 2, 3, \cdots, 3n\}$. Moreover, each element of the friendly index, with the exception of perhaps 1 and $6n-3$, can be obtained with an alternating-bottom labeling f, for which $e_f(1) \geq e_f(0)$.*

Examples 2.7. The vertex labeling on $P_2 \times P_6$, shown in Figure 1, were obtained using the colorings discussed in the proof of Theorem 2.5 and Lemma 2.4.

For the purpose of discussion, we define the following terminology to be used later in this section. Consider an illustration of $P_m \times P_n$ given in Figure 5.

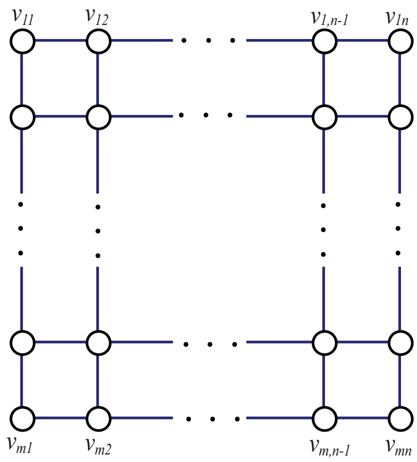

Figure 5: An illustration of $P_m \times P_n$.

We denote the vertex in the ith row and jth column of $P_m \times P_n$ by v_{ij}. Moreover, the vertices $v_{11}, v_{12}, \cdots, v_{1n}$ are defined to be the top vertices, while the vertices $v_{m1}, v_{m1}, \cdots, v_{mn}$ will be named the bottom vertices. The subgraph of $P_m \times P_n$, denoted by R_i, induced by the vertices $\{v_{ij}$:

$j = 1, 2, \cdots, n\}$ is defined to be the i-th row of the grid. Note that the subgraph R_i is simply a path of order n. Given a friendly labeling f defined on a grid, we say that the path R_i has an alternating labeling if $f(v_{i1}) = f(v_{i3}) = \cdots = f(v_{2\lceil \frac{n}{2} \rceil -1,i}) = 1$ and $f(v_{i2}) = f(v_{i4}) = \cdots = f(v_{i,2\lceil \frac{n}{2} \rceil}) = 0$. On the other hand, if f is a labeling such that $f(v_{i1}) = f(v_{i3}) = \cdots = f(v_{2\lceil \frac{n}{2} \rceil -1,i}) = 0$ and $f(v_{i2}) = f(v_{i4}) = \cdots = f(v_{i,2\lceil \frac{n}{2} \rceil}) = 1$, then we say that R_i has alternating-inverse labeling. If $f : V(P_m \times P_n) \to \mathbb{Z}_2$ is a friendly labeling with the property that every path on a grid has alternating labeling or alternating-inverse labeling, then f is the maximum friendly coloring of a grid and $e_f(1) - e_f(0) = m(n-1) + n(m-1)$. It is clear that such a maximum labeling exist for any $n, m \in \mathbb{N}$. Additionally, as mentioned earlier, it has been shown [10] that $P_m \times P_n$ is cordial. We formally state these results in the following Lemma.

Lemma 2.8. *For the grid $P_m \times P_n$ with q edges $\{q - 2\lfloor \frac{q}{2} \rfloor, q\} \subset FI(P_m \times P_n)$*

Lemma 2.9. *If $P_m \times P_n$ has q edges, then $q - 2$ cannot be in the friendly index set.*

Proof. This is a direct consequence of Lemma 1.3. □

Lemma 2.10. *Let $m, n \geq 3$. For the graph $P_m \times P_n$ with q edges, if m and n are odd, then $q - 4 \in FI(P_m \times P_n)$. Otherwise, $q - 4 \notin FI(P_m \times P_n)$.*

Proof. Let $m, n \geq 3$ and suppose f is a labeling for which $e_f(1) = 2$. Note that by Lemma 1.3 the two edges labeled 1 must either be $\{v_{11}v_{21}, v_{11}v_{12}\}$, $\{v_{1n}v_{1,n-1}, v_{1n}v_{2n}\}$, $\{v_{m1}v_{m-1,1}, v_{m1}v_{m2}\}$, or $\{v_{mn}v_{m,n-1}, v_{mn}v_{m-1,n}\}$. Without loss of generality, let f be a binary-vertex labeling for which $\{v_{11}v_{21}, v_{11}v_{12}\}$ edges are labeled 1. For the other edges to be labeled 0 the rest of the vertices must be labeled in the same manner as v_{21} and v_{12} so that $|v_f(0) - v_f(1)| = mn - 2 > 1$; that is, f cannot be friendly.

On the other hand, if $f : V(P_m \times P_n) \to \mathbb{Z}_2$ is a labeling such that $e_f(0) = 2$ then by Lemma 1.3 the two edges labeled 0 must be either $\{v_{11}v_{21}, v_{11}v_{12}\}$, $\{v_{1n}v_{1,n-1}, v_{1n}v_{2n}\}$, $\{v_{m1}v_{m-1,1}, v_{m1}v_{m2}\}$, or $\{v_{mn}v_{m,n-1}, v_{mn}v_{m-1,n}\}$. Suppose f is a binary-vertex labeling for which $\{v_{11}v_{21}, v_{11}v_{12}\}$ are labeled 0 by the induced-edged labeling. For the rest of the edges to labeled all 1, R_i for $i \geq 2$ must have an alternating labeling or alternating-inverse labeling with $f(v_{1i}) = f(v_{21})$ if and only if i is even. In addition, $f(v_{13}) = f(v_{15}) = \cdots, f(1, v_{2\lceil \frac{n}{2} \rceil -1}) = 1$ and the rest of the vertices must be labeled by 0. If m or n are even, it is easy to see that $|v_f(1) - v_f(0)| = 2$, showing that f is not friendly. If, however, m and n are both odd, then $|v_f(1) - v_f(0)| = 1$. □

It remains to be shown whether $\{q - 2i : i = 3, 4, 6, \cdots, \lfloor \frac{q}{2} \rfloor - 1; q = 2mn - m - n\} \subset FI(P_m \times P_n)$. It turns out that such a result is true. In addition, one can find a friendly labeling for which the index is one of these values and the m^{th} row has alternating labeling. Before proceeding to show this, we provide the following useful results.

Lemma 2.11. *For the path P_n of order n and $x \in \{2i-(n-1) : 1 \leq i \leq n-1\}$, there is a friendly labeling $f : V(P_n) \to \mathbb{Z}_2$ for which $e_f(1) - e_f(0) = x$.*

Proof. The result can be easily shown by induction. □

Lemma 2.12. *Let $f : V(P_m \times P_n) \to \mathbb{Z}_2$ be a friendly labeling, with alternating bottom labeling, such that $e_f(1) - e_f(0) = x$. For every $k \in \{0, 2, 4, \cdots, 2n-2\}$ there is a friendly labeling $g : V(P_{m+2} \times P_n) \to \mathbb{Z}_2$ satisfying the following conditions:*

(i) $e_g(1) - e_g(0) = x + k$;

(ii) $g(v) = f(v)$ *for every* $v \in V(P_m \times P_n)$; *and*

(iii) either g or its inverse coloring has alternating-bottom labeling.

Proof. Let $f : V(P_m \times P_n) \to \mathbb{Z}_2$ be a friendly labeling, with alternating-bottom labeling, such that $e_f(1) - e_f(0) = x$.

Case 1. $k \in \{2, 4, \cdots, 2n-2\}$. By Lemma 2.11, there is a labeling $h : V(R_{m+1}) \to \mathbb{Z}_2$ such that $e_h(1) - e_h(0) = k - (n-1)$. Define $g : V(P_{m+2} \times P_n) \to \mathbb{Z}_2$ by

$$g(v_{j,i}) = \begin{cases} f(v_{ij}) & \text{if } v_{ij} \in V(P_m \times P_n); \\ h(v_{ij}) & \text{if } v_{ij} \in V(R_{m+1}); \\ 1 - f(v_{mj}) & \text{if } v_{ij} \in V(R_{m+2}). \end{cases}$$

One readily sees that there are n edges incident to exactly one vertex in R_{m+1} that are labeled 0 and 1. In addition, all $n-1$ edges in R_{m+1} are labeled 1, so that $e_g(1) - e_g(0) = x + k$.

Case 2. $k = 0$. We consider the following two subcases:

Subcase 2a. n is odd.

Let $h : V(R_{m+1}) \to \mathbb{Z}_2$ be defined by $h(v_{m+1,1}) = h(v_{m+1,3}) = \cdots = h(v_{m+1,n-2}) = 1$, $h(v_{m+1,2}) = h(v_{m+1,4}) = \cdots = h(v_{m+1,n-3}) = 0$, and $h(v_{m+1,n-1}) = h(v_{m+1,n}) = 0$. Define $g : V(P_{m+2} \times P_n) \to \mathbb{Z}_2$ by

$$g(v_{j,i}) = \begin{cases} f(v_{ij}) & \text{if } v_{ij} \in V(P_m \times P_n); \\ h(v_{ij}) & \text{if } v_{ij} \in V(R_{m+1}); \\ f(v_{mj}) & \text{if } v_{ij} \in V(R_{m+2}). \end{cases}$$

There are $n-1$ vertices in R_{m+1} adjacent to vertices in R_m and R_{m+2} that are labeled in the same manner. In addition, the vertices $v_{m+1,n}$ and $v_{m+1,n-1}$ are adjacent to each other and are labeled 0. As a result $e_g(0) = e_f(0) + (n-1) + (n-1) + 1 = e_f(0) + 2n - 1$, so that $e_g(1) - e_g(0) = x$.

Subcase 2b. n is even.

We use the same labeling, $g : V(P_{m+2} \times P_n) \to \mathbb{Z}_2$, as in subcase 2a. However, we define $h : V(R_{m+1}) \to \mathbb{Z}_2$ by $h(v_{m+1,1}) = h(v_{m+1,3}) = \cdots = h(v_{m+1,n-5}) = 1$, $h(v_{m+1,2}) = h(v_{m+1,4}) = \cdots = h(v_{m+1,n-6}) = 0$, and $h(v_{m+1,n-4}) = h(v_{m+1,n-3}) = h(v_{m+1,n-2}) = 0$, and $h(v_{m+1,n-1}) = h(v_{m+1,n})$=0. This labeling yields $e_g(1) - e_g(0) = x$. □

Lemma 2.13. *Let $f : V(P_m \times P_n) \to \mathbb{Z}_2$ be a friendly labeling, with alternating bottom labeling, such that $e_f(1) - e_f(0) = x$. Then there is a friendly labeling $g : V(P_{m+2} \times P_n) \to \mathbb{Z}_2$ satisfying the following conditions:*

(i) $e_g(1) - e_g(0) = x + 4n - 2$;

(ii) $g(v) = f(v)$ *for every* $v \in V(P_m \times P_n)$; *and*

(iii) g *has alternating-bottom labeling.*

Proof. Let $f : V(P_m \times P_n) \to \mathbb{Z}_2$ be a friendly labeling, with alternating-bottom labeling, such that $e_f(1) - e_f(0) = x$.
Define $g : V(P_{m+2} \times P_n) \to \mathbb{Z}_2$ by

$$g(v_{ij}) = \begin{cases} f(v_{ij}) & \text{if } v_{ij} \in V(P_m \times P_n); \\ 1 - f(v_{mj}) & \text{if } v_{ij} \in V(R_{m+1}); \\ f(v_{mj}) & \text{if } v_{ij} \in V(R_{m+2}). \end{cases}$$

One readily observes that $e_g(1) - e_g(0) = x + 4n - 2$ and that g has alternating-bottom labeling. □

Theorem 2.14. *For a grid $P_m \times P_n$ with q edges, $\{q - 2i : i = 3, 4, 5, \cdots, \lfloor \frac{q}{2} \rfloor - 1\} \subset FI(P_m \times P_n)$. Moreover, for every $x \in \{q - 2i : i = 3, 4, 5, \cdots, \lfloor \frac{q}{2} \rfloor - 1\}$ there is some friendly labeling, f, with alternating bottom labeling, such that $e_f(1) - e_f(0) = x$.*

Proof. Case 1. $m = 2k$. We proceed by induction on m and fix $n \in \mathbb{N}$. Theorem 2.5 shows that the result is true for the base case $k = 1$.
Assume that $A = \{q_o - 2i : i = 3, 4, 5, \cdots, \lfloor \frac{q_o}{2} \rfloor - 1; q_o = 4k_on - 2k_o - n\} \subset FI(P_{2k_o} \times P_n)$ for some $k_o \in \mathbb{N}$ and that there are friendly labelings that attain these indices for which R_{2k_o} has alternating bottom labeling and $e(1) \geq e(0)$. We consider the graph $P_{2(k_o+1)} \times P_n$ and let $q_1 = q_o + 4n - 2$ be the number of edges in this graph. From the Lemma 2.12, $FI(P_{2(k_o+1)} \times P_n)$ contains $A + \{0, 2, 4, \cdots, 2n-2\}$. Furthermore, there are labeling that attain these indices for which its inverse is an alternating-bottom labeling and $e(1) \geq e(0)$. Both A and $\{0, 2, 4, \cdots, 2n-2\}$ form an arithmetic progression with common difference 2; hence, the set $B = A + \{0, 2, 4, \cdots, 2n - 2\}$ forms an arithmetic progression with common difference 2 for which $\min B = \min A + \min\{0, 2, 4, \cdots, 2n-2\}$ and $\max B = \max A + \max\{0, 2, 4, \cdots, 2n - 2\}$. Therefore, $\{q_1 - 2i : n - 3 \leq i \leq \lfloor \frac{q_1}{2} \rfloor - 1\} \subset FI(P_{2(k_o+1)} \times P_n)$.
Similarly, by Lemma 2.13 $A + 4n - 2 \subset FI(P_{2(k_o+1)} \times P_n)$. Moreover, there are alternating-bottom labelings that attain these indices for which $e(1) \geq e(0)$. Hence, $\{q_1 - 2i : 3 \leq i \leq \lfloor \frac{q_1}{2} \rfloor - 2n\} \subset FI(P_{2(k_o+1)} \times P_n)$. Since $\lfloor \frac{q_1}{2} \rfloor - 2n \geq n - 3$, the result follows immediately.
Case 2. $m = 2k + 1$. It can be shown that the result holds for $P_3 \times P_n$. The argument is similar to the one presented in Case 1. □

By Lemmas 2.8, 2.9, 2.10 and Theorem 2.14, we conclude this paper with our final result:

Theorem 2.15. *Let $m, n \geq 3$. The friendly index set of the grid $P_m \times P_n$, with q edges is*

$$FI(P_m \times P_n) = \begin{cases} \{q - 2i : i = 0, 2, 3, \cdots, \lfloor \frac{q}{2} \rfloor\} & \text{if } m, n \text{ are odd;} \\ \{q - 2i : i = 0, 3, 4, \cdots, \lfloor \frac{q}{2} \rfloor\} & \text{otherwise.} \end{cases}$$

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