# V. Homogeneous Linear Differential Equations

## 1. Linear Differential Equations: Introduction

In Chapter III we learned to solve several types of first-order differential equations. We now turn our attention to differential equations of order higher than one. As one would expect, higher order equations present even greater challenges and we shall consider here only certain subclasses of linear equations, as explained below.

A *linear differential equation of order* $n$ is an equation of the form

$$y^{(n)} + a_1(t)y^{(n-1)} + a_2(t)y^{(n-2)} + \cdots + a_{n-1}(t)y' + a_n(t)y = b(t). \tag{26}$$

Equation (26) is said to be *homogeneous* if $b(t) \equiv 0$. Linear differential equations have diverse applications and are considered by some to be the most important class of differential equations.

The case $n = 1$, that is,

$$y' + a(t)y = b(t),$$

has been dealt with completely in Section 3.7. By contrast, for $n \geq 2$ we shall develop methods for solving (26) only under rather sever restrictions on the coefficients $a_i(t)$ and the function $b(t)$. In the next four sections we consider homogeneous equations with constant coefficients, that is,

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1}y' + a_n y = 0,$$

where the coefficients $a_i$ are fixed constants.

## 2. Second-Order with Constant Coefficients, I

Our present objective is to solve the equation

$$y'' + ay' + by = 0. \tag{27}$$

We begin by observing that it is reasonable to suspect that a solution to (27) may be found amongst functions of the form $y = e^{\lambda t}$. Why?

So let us try this approach. We have

$$\begin{gathered} y = e^{\lambda t}, \quad y' = \lambda e^{\lambda t}, \quad y'' = \lambda^2 e^{\lambda t}, \\ y'' + ay' + by = e^{\lambda t}\big(\lambda^2 + a\lambda + b\big) = 0. \end{gathered}$$

But the last equation holds if and only if

$$\lambda^2 + a\lambda + b = 0. \tag{28}$$

Equation (28) is called the *characteristic equation* associated to (27) and a solution of (28) $\lambda$ is said to be a *characteristic root* of (27). Evidently $y = e^{\lambda t}$ solves (27) if and only if $\lambda$ solves (28).

**2.1 Example** Find the characteristic roots of $y'' + 2y' - 3y = 0$ and verify that they yield solutions to this equation.

So we found a solution to (27) for every root of (28). Since we expect (28) to have two distinct roots $\lambda_1$ and $\lambda_2$ this calculation is expected to yield two distinct solutions to (27), namely

$$y_1 = e^{\lambda_1 t} \quad \text{and} \quad y_2 = e^{\lambda_2 t}. \tag{29}$$

But observe that if $y_1$ and $y_2$ are any solutions of (27) (not necessarily (29)), and if

$$y = c_1 y_1 + c_2 y_2, \tag{30}$$

for some constants $c_1$ and $c_2$, then (30) is also a solution of (27). Indeed,

$$y' = c_1 y_1' + c_2 y_2', \quad y'' = c_1 y_1'' + c_2 y_2'',$$

whence

$$\begin{aligned} y'' + ay' + by &= c_1 y_1'' + c_2 y_2'' + a\big(c_1 y_1' + c_2 y_2'\big) + b\big(c_1 y_1 + c_2 y_2\big) \\ &= c_1\big(y_1'' + ay_1' + by_1\big) + c_2\big(y_2'' + ay_2' + by_2\big) \\ &= 0. \end{aligned}$$

**2.2 Example** Since $e^{-3t}$ and $e^t$ are solutions of the equation $y'' + 2y' - 3y = 0$ then so is any function of the form

$$y = c_1 e^{-3t} + c_2 e^t,$$

where $c_1$ and $c_2$ are constants. Actually, we often opt for the more convenient

$$y = ae^{-3t} + be^t \qquad [a, b - \text{constants}].$$

**2.3 Linear Combinations** Functions of the form (30) are called *linear combinations* of $y_1$ and $y_2$. We showed that any linear combination of solutions of (27) is again a solution of (27). This is a very important and fruitful idea, so let us now consider what really made our argument work:

- Does the order matter?
- Do we need the coefficients to be constant?
- What if the equation is not linear?
- What if the equation is not homogeneous?

It is now plain that the same argument yields the following general fact.

**Proposition.** Any linear combination of solutions of a homogeneous linear differential equation is also a solution of that equation.

*Proof.* Exercise.

Back to (27). We found that if the characteristic equation (28) has two distinct roots (as expected) then functions $y_1$ and $y_2$ given in (29) and their linear combinations (30) are solutions of (27). There remains the question of whether there are other solutions of (27). The answer is given by the following.

**2.3 Proposition** If the characteristic equation (28) has two distinct roots $\lambda_1$ and $\lambda_2$, then every solution of (27) is given by

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

for some constants $c_1$ and $c_2$.

We will not give a proof of this proposition.

**2.4 Example** The set of solutions of the differential equation $y'' + 2y' - 3y = 0$ is given by $y = ae^{-3t} + be^{t}$, where $a$ and $b$ are arbitrary constants.

**2.5 Example** Solve the equation $y'' - 4y' - y = 0$.

**2.6 Example** Solve the equation $2y'' - 3y = 0,\ y(0) = 1,\ y'(0) = 3$.

**2.7 Exercises** Solve the following differential equations.

(a) $y'' - 2y' - 3y = 0, \quad y(0) = 2,\ y'(0) = 1$

(b) $3y'' + y' - 2y = 0, \quad y(0) = 1,\ y'(0) = 1$

(c) $2y'' - y = 0, \quad y(0) = 1,\ y'(0) = 1$

(d) $3y'' + y' = 0$

(e) $y'' + 2y' - 5y = 0$

## 3. Second-Order with Constant Coefficients, II

In Section 2 we dealt with the typical case of the equation $y'' + ay' + by = 0$ – whose characteristic equation (28) had two distinct roots. In this section we complete the job by considering the remaining two cases: characteristic equation has only one root (double root) or no (real) roots. We begin with the double root case,

$$\lambda^2 + a\lambda + b = (\lambda - \lambda_0)^2 \quad [a = -2\lambda_0,\ b = \lambda_0^2]. \tag{31}$$

In this case we already know from Section 2 that any function of the form $ce^{\lambda_0 t}$, where $c$ is an arbitrary constant, is a solution. Since no further solutions are forthcoming from (28)/(31) one might suspect that this captures all possible solutions of (27). This is false. To see this consider $y = te^{\lambda_0 t}$:

$$y' = e^{\lambda_0 t} + \lambda_0 te^{\lambda_0 t}, \quad y'' = \lambda_0 e^{\lambda_0 t} + \lambda_0\big(e^{\lambda_0 t} + \lambda_0 te^{\lambda_0 t}\big) = \lambda_0^2 te^{\lambda_0 t} + 2\lambda_0 e^{\lambda_0 t};$$

$$\begin{aligned} y'' + ay' + by &= e^{\lambda_0 t}\big[t(\lambda_0^2 + a\lambda_0 + b) + (2\lambda_0 + a)\big] \\ &= e^{\lambda_0 t}\big[t \cdot 0 + 0\big] \\ &= 0. \end{aligned}$$

At this point we know that functions $e^{\lambda_0 t}$ and $te^{\lambda_0 t}$ "work" and hence any linear combination of these functions is also a solution. In fact, this exhausts all possible solutions of (27) in the present case.

**3.1 Proposition** If the characteristic equation has a double root $\lambda_0$ ((31) holds), then every solution of (27) is given by

$$y = c_1 e^{\lambda_0 t} + c_2 te^{\lambda_0 t} = (c_1 + c_2 t)e^{\lambda_0 t},$$

where $c_1$ and $c_2$ are constants.

We will not give a proof of this proposition either.

**3.2 Example** Solve the equation $y'' - y' + \frac{1}{4}y = 0,\ y(0) = 0,\ y'(3) = 1.$

Finally, we consider the case where the characteristic equation has no (real) roots, viz:

$$\lambda^2 + a\lambda + b = (\lambda + a/2)^2 + b - a^2/4 \quad \text{and} \quad b - a^2/4 > 0. \tag{32}$$

This case is more complicated. In fact, we have no solutions as yet.

Our method for solving (27) in this case depends on the following manipulation of (32). It will make things more transparent if we first illustrate this manipulation on a concrete example.

**3.3 Example** Consider the equation $\lambda^2 - \lambda + 1 = 0$.

The general case of the preceding computation is then as follows. Given (32), write

$$\begin{aligned}\lambda &= \frac{-a \pm \sqrt{a^2 - 4b}}{2} = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4b}}{2} \qquad [a^2 - 4b < 0] \\ &= -\frac{a}{2} \pm \frac{\sqrt{(4b - a^2)(-1)}}{2} \\ &= -\frac{a}{2} \pm \frac{\sqrt{4b - a^2}}{2}\sqrt{-1} \qquad [4b - a^2 > 0] \\ &= \alpha \pm \omega\sqrt{-1}.\end{aligned} \tag{33}$$

The utility of this expression lies in the fact that

**3.4 Claim** Functions $y_1 = e^{\alpha t}\sin(\omega t)$ and $y_2 = e^{\alpha t}\cos(\omega t)$ are solutions of (27).

To verify this claim one simply substitutes $y_i$ into (27) and simplifies the resulting expressions. You are asked to perform this calculation as an exercise.

In view of the preceding discussion no one will be surprised at what comes next, or at the fact that it will not be proved.

**3.5 Proposition** If the characteristic equation has no roots ((32) holds), then every solution of (27) is given by

$$y = c_1 e^{\alpha t}\sin(\omega t) + c_2 e^{\alpha t}\cos(\omega t) = \big(c_1\sin(\omega t) + c_2\cos(\omega t)\big)e^{\alpha t},$$

where $\alpha$ and $\omega$ are as in (33) and $c_i$ are constants.

**3.6 Example** Solve the equation $y'' - y' + y = 0$.

**3.7 Example** Solve the equation $y'' + y = 0$.

## 3.8 Exercises

1. Solve the following differential equations.

   (a) $y'' - 6y' + 9y = 0, \quad y(0) = y'(0) = 1$

   (b) $y'' + 2y' + 3y = 0, \quad y(0) = 1,\ y'(0) = 2$

   (c) $2y'' + 3y = 0$

   (d) $2y'' - 3y = 0$

   (e) $2y'' + 2y' + \frac{1}{2}y = 0$

   (f) $y'' - 5y' + 7y = 0$

2. Verify Claim 3.4.

## 4. Third-Order with Constant Coefficients

In this section we solve the equation

$$y''' + ay'' + by' + cy = 0. \tag{34}$$

Just as in the second-order case it is reasonable to try $y = e^{\lambda t}$: [Why?]

$$y = e^{\lambda t}, \quad y' = \lambda e^{\lambda t}, \quad y'' = \lambda^2 e^{\lambda t}, \quad y''' = \lambda^3 e^{\lambda t},$$
$$y''' + ay'' + by' + cy = e^{\lambda t}\bigl(\lambda^3 + a\lambda^2 + b\lambda + c\bigr) = 0.$$

Evidently $y = e^{\lambda t}$ solves (34) if and only if $\lambda$ solves the associated characteristic equation

$$p(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c = 0. \tag{35}$$

The situation is analogous to that of Sections 2 and 3, except for the number of characteristic roots, i.e., roots of (35). More precisely, we have the following four possibilities:

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \qquad [\lambda_i \neq \lambda_j]; \tag{36}$$

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)^2 \qquad [\lambda_i \neq \lambda_j]; \tag{37}$$

$$p(\lambda) = (\lambda - \lambda_0)^3; \tag{38}$$

$$\begin{aligned} p(\lambda) &= \bigl(\lambda - \lambda_0\bigr)\bigl(\lambda^2 + A\lambda + B\bigr) \qquad [\lambda^2 + A\lambda + B > 0] \\ &= \bigl(\lambda - \lambda_0\bigr)\bigl(\lambda - (\alpha + \omega\sqrt{-1})\bigr)\bigl(\lambda - (\alpha - \omega\sqrt{-1})\bigr). \end{aligned} \tag{39}$$

The analogy with the second-order case extends also to the dependence of a general solution of (34) on the factorization of the characteristic polynomial (36)-(39). Thus obvious modifications to our development in Sections 2 and 3 lead to the following conclusion.

**4.1 Proposition** Depending on the case (36)-(39), the general solution of (34) is given by

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t}, \tag{40}$$

$$y = c_1 e^{\lambda_1 t} + \big(c_2 + + c_3 t\big) e^{\lambda_2 t}, \tag{41}$$

$$y = \big(c_1 + + c_2 t + c_3 t^2\big) e^{\lambda_0 t}, \tag{42}$$

$$y = c_1 e^{\lambda_0 t} + \big(c_2 \sin(\omega t) + c_3 \cos(\omega t)\big) e^{\alpha t}, \tag{43}$$

respectively, where the constants $c_1$, $c_2$, and $c_3$ are arbitrary.

Another analogy with Sections 2 and 3 is that we only verify that the stated solution works, but not the fact that it encompasses all possible solutions of (34).

*Proof [stated solution works].* Recall that we have shown in Proposition 2.3 that if $y_1$ and $y_2$ are any two solutions of (34) then so is their linear combination $c_1y_1+c_2y_2$. Another application of that proposition shows that if $y_3$ is another solution then so is the linear combination

$$c_1y_1 + c_2y_2 + c_3y_3 = (c_1y_1 + c_2y_2) + c_3y_3. \qquad \text{[Why?]}$$

But we already know that each of the functions $e^{\lambda_i t}$ is a solution. Therefore, we only need to show that:

(37) $\Rightarrow$ $te^{\lambda_2 t}$ is a solution;

(38) $\Rightarrow$ $te^{\lambda_0 t}$ and $t^2 e^{\lambda_0 t}$ are solutions;

(39) $\Rightarrow$ $e^{\alpha t}\sin(\omega t)$ and $e^{\alpha t}\cos(\omega t)$ are solutions.

As we already mentioned, the required computations are completely analogous to the corresponding computations in Section 3, and are left to you as exercise.

**4.2 Example** Solve $y''' + 6y'' + 8y' + 3y = 0$.

**4.3 Example** Solve $y''' - 5y'' + y' - 5y = 0$.

**4.4 Example** Solve $y''' + 6y'' + 12y' + 8y = 0,\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 2.$

### 4.5 Exercises

1. Solve the following differential equations.

   (a) $y''' - 3y'' + 3y' - y = 0$

   (b) $y''' - y'' - 7y' + 7y = 0$

   (c) $y''' + 7y'' + 15y' + 9y = 0$

   (d) $4y''' - 8y'' + 5y' - y = 0$

   (e) $y''' + y' = 0$

   (f) $2y''' - y'' - 8y' + 4y = 0,\ y(0) = -1,\ y'(0) = 1,\ y''(0) = 0$

   (g) $4y''' + 8y'' + 41y' + 37y = 0$

   (h) $y''' + \frac{3}{2}y'' + \frac{3}{4}y' + \frac{1}{8}y = 0$ *Hint:* This is the case (38)/(42).

2. Complete the proof of Proposition 4.1.